University of Malta

Junior College

Subject: **Advanced Applied Mathematics**
Date: **June 2011**
Time: **13.00 - 16.00**

End of Year Test

Worked Solutions

Question 1

(i) The weight = 40g = 400 N vertically downwards

$\therefore\ W = \underline{-400\,\mathbf{k}}$ N

(ii) At equilibrium $F_1 + F_2 + F_3 + F_4 = -W = -<0,\ 0, -400>$

Let $\mathbf{F_4}$ = < a, b, c > i.e. $\mathbf{F_4}$ = a**i** +b**j** + c**k**

Hence on substituting we get

80**i** + 20**j** + 100**k** + 60**i** – 40**j** + 80**k** – 50**i** – 100**j** + 80**k** + a**i** +b**j** + c**k** = 400**k**

Equate coefficients of **i**: 80 + 60 – 50 + a = 0 $\Rightarrow$ a = – 90

Equate coefficients of **j**: 20 – 40 – 100 + b = 0 $\Rightarrow$ b = 120

Equate coefficients of **k**: 100 + 80 + 80 + c = 400 $\Rightarrow$ c = 140

$\therefore$ $\mathbf{F_4}$ = – 90**i** + 120**j** + 140**k**

Magnitude of $\mathbf{F_4}$ = $\sqrt{(-90)^2 + (120)^2 + (140)^2} = \sqrt{42100} = \underline{205\text{N}}$

(iii) Using $\cos\theta = \dfrac{a \cdot b}{|a||b|}$, where θ is the angle between the forces $\mathbf{F_1}$ and $\mathbf{F_4}$

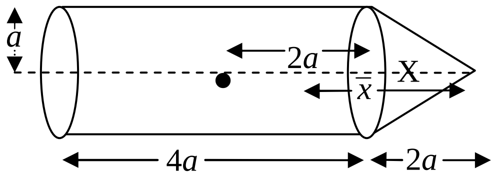
$$\therefore\ \cos\theta\ =\ \frac{<80, 20, 100> \cdot <-90, 120, 140>}{\sqrt{80^2 + 20^2 + 100^2}\sqrt{(-90)^2 + 120^2 + (140)^2}}$$

$$\cos\theta = \frac{-7200 + 2400 + 14000}{\sqrt{16800}\sqrt{42100}} = 0.3459$$

Hence $\theta = \underline{69.76^0}$

Question 2

(i)



Volume of cylinder = $\pi r^2 h = \pi(a)^2(4a) = 4\pi a^3$

Volume of cone = $\frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(a)^2(2a) = \frac{2}{3}\pi a^3$

Let M be the mass of the cone,

then density of cone = $\rho_{\text{cone}} = \frac{M}{V_{\text{cone}}} = \frac{M}{\frac{2}{3}\pi a^3} = \frac{3M}{2\pi a^3}$

Since density of cone = 2(density of cylinder)

then $\rho_{\text{cylinder}} = \frac{1}{2}\rho_{\text{cone}} = \frac{1}{2}\left(\frac{3M}{2\pi a^3}\right) = \frac{3M}{4\pi a^3}$

But $\rho_{\text{cylinder}} = \frac{3M}{4\pi a^3} = \frac{M_{\text{cylinder}}}{V_{\text{cylinder}}} = \frac{M_{\text{cylinder}}}{4\pi a^3}$

$\Rightarrow M_{\text{cylinder}} = \frac{3M}{4\pi a^3}\left(4\pi a^3\right) \quad = \underline{3M}$

(ii)

By symmetry the centre of mass lies on the axis of symmetry of the model.

If distances are measured from the vertex of the cone, then

Centre of mass of a right circular cone of height h is $\frac{3}{4}h$.

$\therefore$ it is $\frac{3}{4}(2a) = \frac{3a}{2}$. In the diagram it is marked with X.

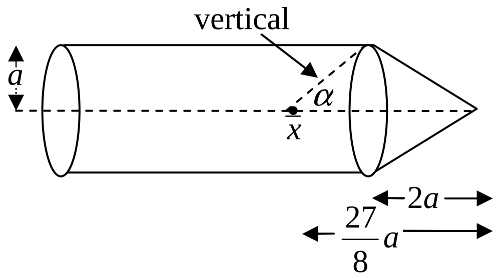
Centre of mass of a cylinder of height h is $\frac{h}{2}$.

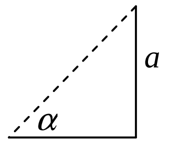
$\therefore$ it is $2a + \frac{1}{2}(4a) = 2a + 2a = 4a$. It is marked with $\bullet$

The centre of mass is found by using the fact that the sum of the moments of the weights of the constituent particles about any line is equal to the moment of the resultant weight about the same line.

Thus $\bar{x}=\dfrac{\sum m_i x_i}{\sum m_i}=\dfrac{M\left(\frac{3}{2}a\right)+3M(4a)}{M+3M}=\dfrac{\frac{3}{2}a+12a}{4}=\underline{\dfrac{27}{8}a}$

(iii)





$\dfrac{27}{8}a-2a=\dfrac{11}{8}a$

By referring to this right angle triangle,

$\tan\alpha=\dfrac{a}{\frac{11}{8}a}=\dfrac{8}{11}$. Hence

$\alpha=\tan^{-1}\left(\dfrac{8}{11}\right)=\underline{36.03^0}$

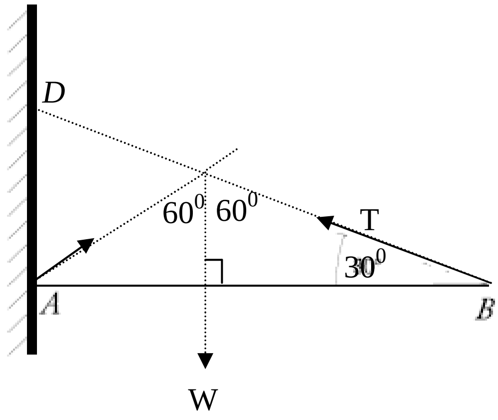
(iv)

By taking moments at vertical axis through point of suspension, we get

$4M\left(\dfrac{11}{8}a\right)=M'(2a)$

$\Rightarrow \quad M'=\dfrac{4M\left(\frac{11}{8}a\right)}{(2a)}=\underline{\dfrac{11}{4}M}$.

Question 3



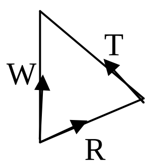
(i)

Using the 3 – force result, the reaction at *A* passes through the point of intersection of the lines of action of the weight W and tension T.

$\therefore$ R makes 30^0 with *AB*.

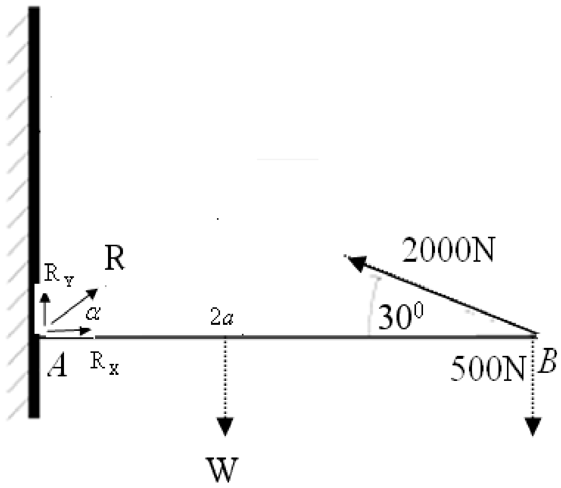
Also triangle *DAC* is a triangle of forces and from its geometry we see that it is an equilateral triangle because angle *ACD* = $180^0 - 60^0 - 60^0 = 60^0$.

Hence we have



$\Rightarrow$ $\underline{\text{ T = R = W}}$

(ii)



At equilibrium, sum of clockwise moments = sum of anticlockwise moments:

$$Wa + 500(2a) = 2000\cos 60^0(2a)$$

This reduces to

$$Wa + 1000a = 2000a$$

$$\Rightarrow Wa = 1000a$$

or W = $\underline{1000\text{ N}}$

(iii) At equilibrium, resolving forces in the horizontal direction we get:

$$R_x = 2000\cos 30^0$$

$$\Rightarrow R_x = \underline{1000\sqrt{3}\text{N}}$$

Resolving forces in a vertical direction, we get;

$$R_y - W + 2000\cos 60^0 - 500 = 0$$

$$R_y - 1000 + 1000 - 500 = 0$$

$$\Rightarrow R_y = \underline{500\text{ N}}$$

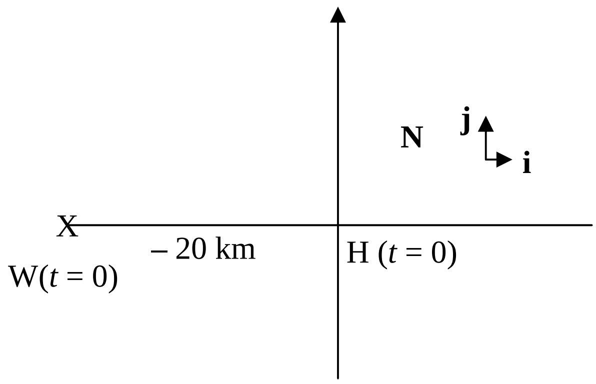
With respect to the diagram above, $\tan\alpha = \dfrac{R_y}{R_x} = \dfrac{500}{1000\sqrt{3}} = \dfrac{1}{2\sqrt{3}}$

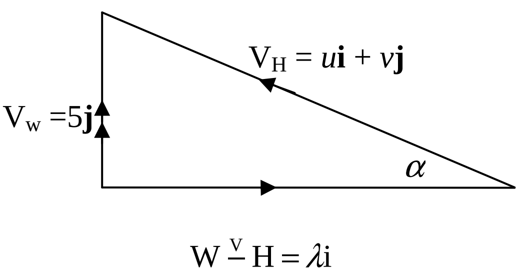
$$\Rightarrow \alpha = \tan^{-1}\left(\frac{1}{2\sqrt{3}}\right) = \underline{16.1^0}$$

The magnitude of the reaction is R = $\sqrt{R_x{}^2 + R_y{}^2} = \sqrt{\left(1000\sqrt{3}\right)^2 + (500)^2}$

$$\therefore R = \sqrt{30000000 + 250000} = \underline{1802.78\text{ N}}$$

Question 4





(i)

At time t, $r_H(t)$ = position vector of helicopter = $(u\mathbf{i} + v\mathbf{j})t$ = $\underline{ut\mathbf{i} + vt\mathbf{j}}$

At time t, $r_W(t)$ = position vector of windsurfer = initial position + velocity after time t = $\underline{-20\mathbf{i} + 5t\mathbf{j}}$

(ii) When the helicopter reaches the windsurfer, $r_H(t) = r_W(t)$

$\therefore$ by (i) above $ut\mathbf{i} + vt\mathbf{j} = -20\mathbf{i} + 5t\mathbf{j}$

By equating coefficients of **i** and **j**

$ut = -20$ and $vt = 5t$ or $v = 5$, since $t \neq 0$…(i)

But we are given that the maximum speed is 100 km h^{-1}

$\therefore \sqrt{u^2 + v^2} = 100$, since $V_H = u\mathbf{i} + v\mathbf{j}$

$\Rightarrow u^2 + 5^2 = 10000$ by using (i)

Hence $u^2 = 10000 - 25 = 9975$

i.e. $u = \sqrt{9975} = \pm 99.87 \text{km h}^{-1}$

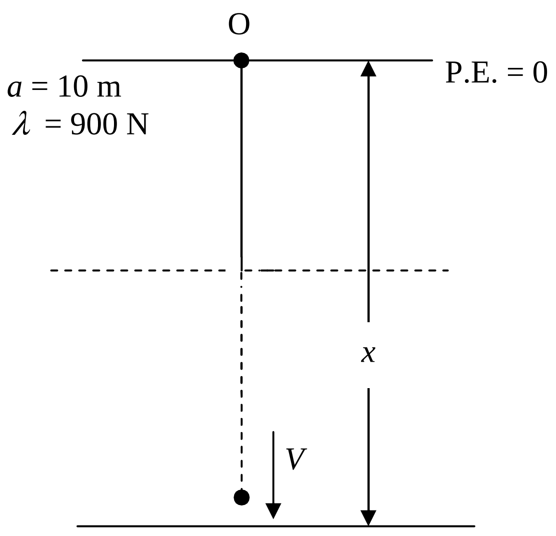
Substituting this value in $ut = -20$, we get $t = \dfrac{-20}{-99.87} = \underline{0.2\,\text{hr}} = \underline{12\,\text{min}}$

With respect to the diagram, $\sin\alpha = \dfrac{|V_W|}{|V_H|} = \dfrac{5}{100} = \dfrac{1}{20}$

$\therefore \alpha = \sin^{-1}\left(\dfrac{1}{20}\right) = 2.9^0$

Hence the bearing of the course = $270^0 + 2.9^0 = \underline{272.9^0}$

Question 5



(i) Initially E.P.E. + P.E. + K.E. = 0

At a general position x m below O, we have:

P.E. = mgh = 60 (10) ($-x$) = $-600x$

$$\text{K.E.} = \frac{1}{2}mv^2 = \frac{1}{2}(60)v^2 = 30v^2$$

$$\text{E.P.E.} = \frac{\lambda x^2}{2a} = \frac{900(x-10)^2}{2(10)} = 45(x-10)^2 = 45(x^2-20x+100)$$

Applying conservation of energy, we have:

$0 = -600x + 30v^2 + 45(x^2 - 20x + 100)$

On dividing throughout by 15, we get:

$0 = -40x + 2v^2 + 3(x^2 - 20x + 100)$

Expanding: $0 = -40x + 2v^2 + 3x^2 - 60x + 300$

Bringing v^2 subject of the equation:

$2v^2 = 40x - 3x^2 + 60x - 300 = 100x - 3x^2 - 300$

$$v^2 = \underline{50x - \frac{3}{2}x^2 - 150}$$

(ii) The maximum length of the rope occurs when $v = 0$

$$\therefore -\frac{3}{2}x^2 + 50x - 150 = 0$$

$$-3x^2 + 100x - 300 = 0$$

$$\therefore\ x = \frac{-100 \pm \sqrt{100^2 - 4(-3)(-300)}}{2(-3)} = \frac{-100 \pm \sqrt{10000 - 3600}}{-6}$$

$$\Rightarrow\ x = \frac{-100 \pm \sqrt{6400}}{-6} = \frac{-100 \pm 80}{-6} = -30 \text{ or } \frac{20}{6}$$

Hence the maximum length of the rope is <u>30 m</u>, since the other length is not possible because the original length is 10 m.

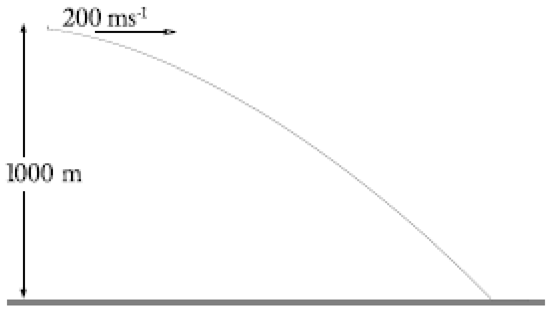
(iii) The work done against air resistance = K.E. that the particle would have if there were no air resistance.

From the above equation when $x = 25$ m

$$v^2 = -\frac{3}{2}(25)^2 + 50(25) - 150 = 162.5\ \text{ms}^{-1}$$

$$\therefore\ \text{K.E.} = \text{Work Done} = \frac{1}{2}(60)(162.5) = \underline{4875\ \text{J}}$$

Question 6



(i)

$\updownarrow$

$s = 1000 \text{ m}$
$u = 0$
$a = 10 \text{ ms}^{-2} \downarrow$
$t = ?$

$\therefore$ Using the formula $s = ut + \frac{1}{2}at^2$

we get $1000 = 0 + \frac{1}{2}(10)t^2$

$\Rightarrow \ t^2 = \dfrac{2000}{10} = 200$ or $t = \sqrt{200} = \underline{14.14 \text{ s}}$

$\leftrightarrow$

Applying the formula $x = ut = 200\ (14.14) = \underline{2828 \text{ m}}$

(ii) At time t : $x = ut = 200\,t$. $\Rightarrow t = \dfrac{x}{200}$

$s = y$
$u = 0$
$a = -10 \text{ ms}^{-2} \uparrow$
$t = \dfrac{x}{200}$

$\therefore$ Using the formula $s = ut + \frac{1}{2}at^2$

we get $y = 0 + \frac{1}{2}(-10)t^2 = -5t^2$

Hence the coordinates are $\underline{\left(200t, -5t^2\right)}$

(iii) From (ii) we get $y = -5t^2 = -5\left(\dfrac{x}{200}\right)^2 = \underline{-\dfrac{x^2}{8000}}$

(iv) From (iii) $y = -\dfrac{x^2}{8000}$

$\dfrac{dy}{dx} = -\dfrac{2x}{8000} = -\dfrac{x}{4000}$, by differentiating with respect to x

(Direction of the velocity vector is the same as the direction of motion)

When the bomb hits the target, $x = 2828$ m from part (i)

$\therefore \dfrac{dy}{dx} = -\dfrac{2828}{4000} = -0.7071$

$\Rightarrow \tan\alpha = -0.7071$ (negative sign implies downward direction)

Hence $\alpha = \tan^{-1} 0.7071 = \underline{35.26^0}$ to the horizontal

Question 7

(i) $v = 63 \text{ kmh}^{-1} = \dfrac{63(1000)\,\text{m}}{60(60)\,\text{s}} = 17.5\,\text{ms}^{-1}$ $\quad m = 40$ tonnes = 40000 kg

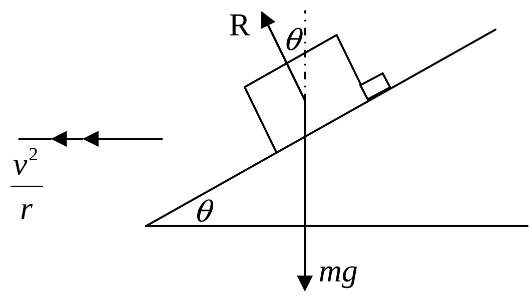
$r = 1.25$ km = 1250 m

Force exerted on rail = centripetal force = $\dfrac{mv^2}{r} = \dfrac{40000(17.5)^2}{1250}$

$\underline{9800 \text{ N}}$

and it points outwards, away from the centre of the circular path described.

(ii)



Resolving vertically:

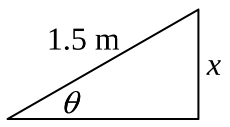
$\updownarrow$ $\mathrm{R}\cos\theta = mg = 40000(10) = 400000$ ….(a)

Resolving horizontally: $\mathrm{F} = ma = \dfrac{mv^2}{r}$

$$\Rightarrow \mathrm{R}\sin\theta = \frac{mv^2}{r} = 9800 \text{ from (i) above … (b)}$$

(b) $\div$ (a): $\dfrac{\mathrm{R}\sin\theta}{\mathrm{R}\cos\theta} = \dfrac{9800}{400000} = 0.0245$

$$\Rightarrow \tan\theta = 0.0245 \quad \text{or} \quad \theta = \tan^{-1}(0.0245) = 1.403^0$$



By referring to the diagram:

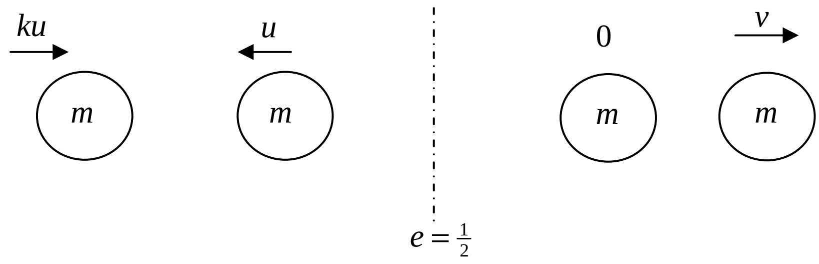
$\sin\theta = \dfrac{x}{1.5} \quad \Rightarrow \; x = 1.5\sin\theta$

$\therefore \; x = 1.5\sin(1.403^0) = 0.0367\,\mathrm{m}$

Hence x = <u>36.7 mm</u>

Question 8

(a)



Applying the conservation of linear momentum:

$mku - mu = mv \qquad \Rightarrow \; ku - u = v \;$ or $\; v = u(k-1)$…(i)

The coefficient of restitution $e = \dfrac{\text{Separation speed}}{\text{Approach speed}} = \dfrac{v - 0}{ku - (-u)}$

$$\Rightarrow \frac{1}{2} = \frac{v}{ku + u} \quad \text{or} \quad 2v = u(k+1) \text{ …(ii)}$$

(ii) $\div$ (i) $\quad \dfrac{2v}{v} = \dfrac{u(k+1)}{u(k-1)} \quad \Rightarrow 2 = \dfrac{k+1}{k-1}$

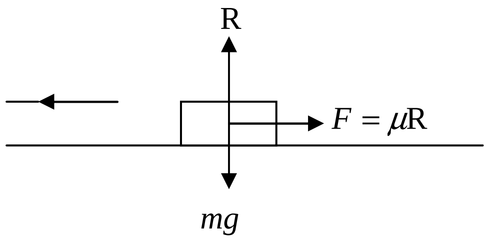
$k + 1 = 2k - 2$

$3 = k$

Hence the ratios of the speed before impact = $\underline{3.}$

(b) (i) After the particle is set in motion, there are no external forces. Thus by the principle of conservation of momentum on the system as a whole, the net impulse must be zero.

$\Rightarrow$ Impulse of friction = Initial impulse given = I.



(ii) Resolving vertically R = mg

But $F = \mu R = \underline{\mu mg}$

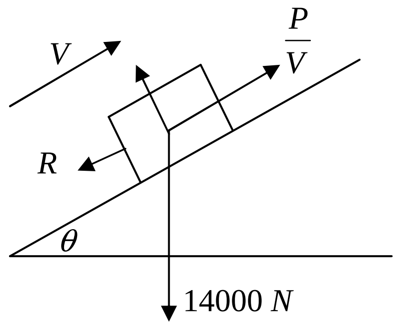
(iii) Impulse = Ft = $\Rightarrow$ $I = Ft = \mu mgt$

$$\therefore t = \underline{\frac{I}{\mu mg}}$$

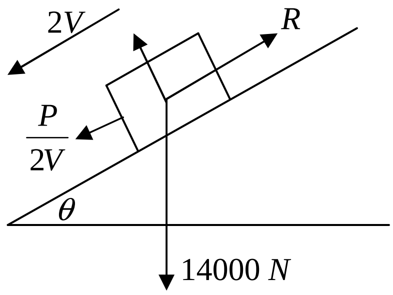
Question 9

(i)

Car ascending



Car descending



Since we have steady speed, then we have equilibrium in both cases.

Considering the case when the car is ascending:

$$\frac{P}{V} = \mathrm{R} + 14000\sin\theta = \mathrm{R} + 14000\left(\frac{1}{56}\right) = \mathrm{R} + 250$$

Since P = 10 kW = 10000 W, then

$$\frac{10000}{V} = \mathrm{R} + 250 \text{ ….(a)}$$

Consider the case when the car is descending:

$$\frac{P}{2V} = \mathrm{R} - 14000\sin\theta = \mathrm{R} - 14000\left(\frac{1}{56}\right) = \mathrm{R} - 250$$

Since P = 10 kW = 10000 W, then

$$\frac{10000}{2V} = \mathrm{R} - 250$$

$$\frac{5000}{V} = \mathrm{R} - 250\text{...(b)}$$

(ii) (a) – (b), we get

$$\frac{10000}{V} - \frac{5000}{V} = 250 + 250$$

$$\frac{5000}{V} = 500$$

$$\Rightarrow \quad 5000 = 500V \quad \text{or } \underline{V = 10\text{ms}^{-1}}$$

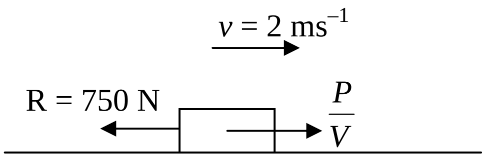
Substituting this value of V in (a) we get:

$$\frac{10000}{10} = \text{R} + 250$$

$$1000 = \text{R} + 250$$

$$\Rightarrow \quad \underline{\text{R} = 750\text{ N}}$$

(iii)



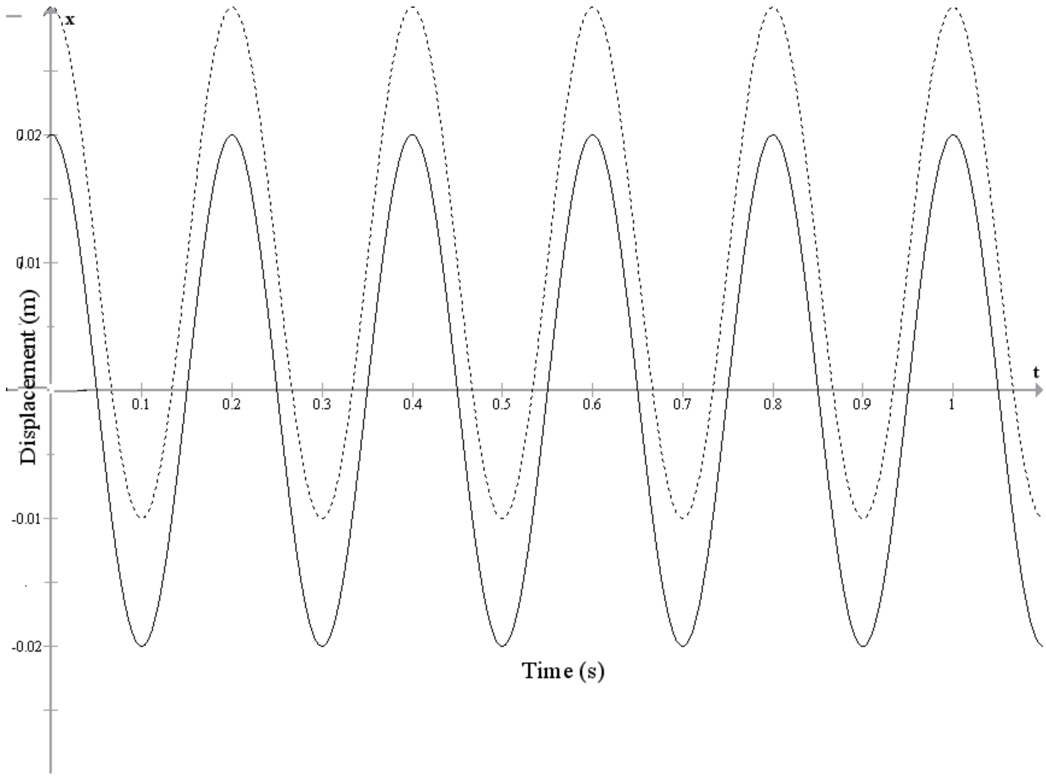
Applying $F = ma$, we have

$$\frac{P}{V} - \text{R} = 1400a$$

$$\Rightarrow \frac{10000}{2} - 750 = 1400a$$

$$5000 - 750 = 1400a \quad \text{or} \quad a = \frac{5000 - 750}{1400} = \underline{3.04\text{ms}^{-2}}$$

Question 10



(i) The graph of $x = a\cos\omega t$ is the black continuous one, while the dotted black one is the graph of $x = a\cos\omega t + d$.

By comparison the maximum value of the black continuous one is 0.02, while the dotted one is 0.03.

This implies that d = 0.03 – 0.02 = $\underline{0.01}$

(ii) $x = a\cos\omega t + 0.01$

Differentiating it we get $\dot{x} = -\omega a\sin\omega t$

Differentiating it another time: $\underline{\ddot{x}} = \omega^2 a\cos\omega t = \underline{\omega^2(x - 0.01)}, \ldots (1)$

since $a\cos\omega t = x - 0.01$

Let $X = x - 0.01$

On differentiating it we get: $\dot{X} = \dot{x}$

Differentiating it another time: $\ddot{X} = \ddot{x}$

$\therefore$ equation (1) becomes $\underline{\ddot{X} = \omega^2 X}$.

This is the basic equation of S.H.M.

(iii) By referring to the graph of the black continuous one:

a (amplitude) = <u>0.02</u>,

T (the period) = <u>0.2 s</u> – time taken for one complete oscillation

But $T = \dfrac{2\pi}{\omega} \Rightarrow \omega = \dfrac{2\pi}{\mathrm{T}} = \dfrac{2\pi}{0.2} = \underline{10\pi \text{ rad/s}}$

The frequency $f = \dfrac{1}{T} = \dfrac{1}{0.2} = \underline{5\,\text{Hz}}$ or 5 oscillations every second.

(iv) Since $x = a\cos\omega t + d$,

Substituting the values obtained we get: $x = 0.02\cos 10\pi t + 0.01$

When x = o, we get $0.02\cos 10\pi t + 0.01 = 0$

Or $\cos 10\pi t = -\dfrac{0.01}{0.02} = -\dfrac{1}{2}$

$\Rightarrow \quad 10\pi t = \dfrac{2\pi}{3}$, the negative sign implies that we have obtuse angle

$\Rightarrow \quad t = \dfrac{2\pi}{3(10\pi)} = \dfrac{2}{30} = \underline{\dfrac{1}{15}\text{ s}}$