University of Malta

Junior College

Subject:	**Advanced Applied Mathematics**
Date:	**June 2012**
Time:	**9.00 - 12.00**

End of Year Test

Worked Solutions

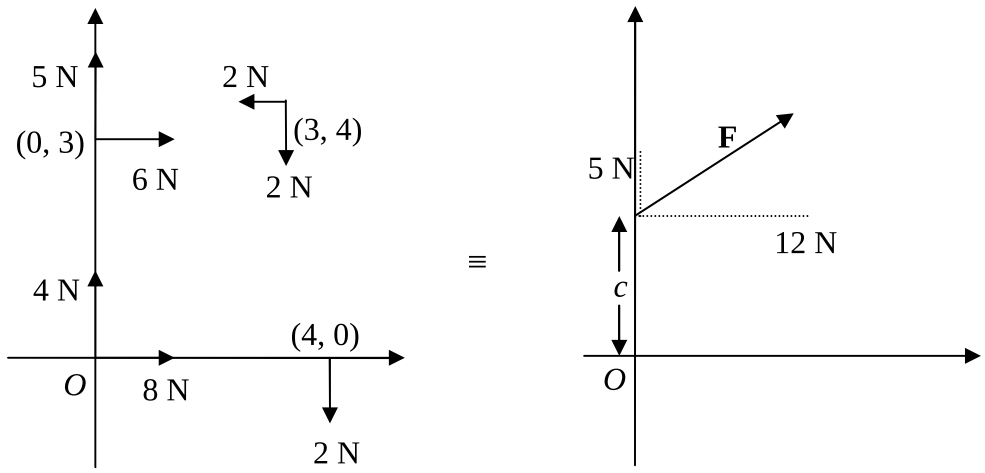
Question 1

(i) $\mathbf{F} = \sum \mathbf{F}_i = (8 + 6 - 2)\mathbf{i} + (4 + 5 - 2 - 2)\mathbf{j} = (12\mathbf{i} + 5\mathbf{j})$ N

$\Rightarrow$ magnitude of $\mathbf{F} = |\mathbf{F}| = \sqrt{12^2 + 5^2} = \underline{13\text{ N}}$

Let θ be the angle that **F** makes with the horizontal, then

$\tan\theta = \frac{5}{12} \Rightarrow \theta = \tan^{-1}\left(\frac{5}{12}\right) = \underline{22.6^0}$



(ii) Clockwise moment about O: $6(3) + 2(4) + 2(3) - 2(4) = 12c$

$\Rightarrow 24 = 12c$ i.e. $c = \underline{2\text{ m}}$

Hence the line of action of **F** passes through the point (0, 2).

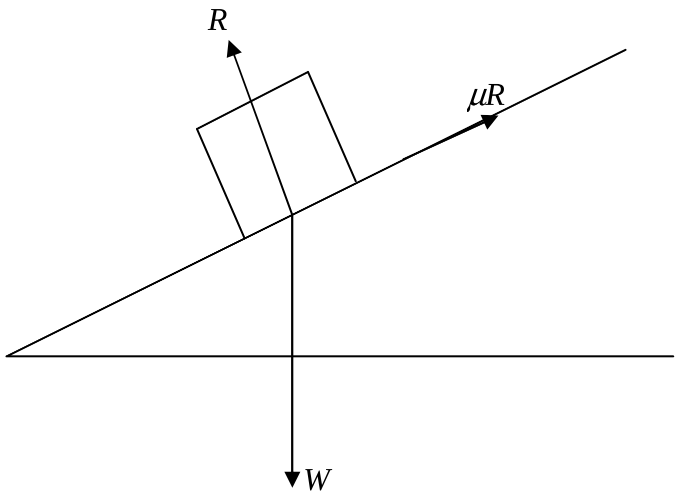
The equation of the line is $y = \frac{5}{12}x + 2$ or $\underline{12y = 5x + 24}$

(iii) From part (ii), the clockwise moment about O is 12(2) = 24 Nm

Hence the magnitude of $C = |C| = \underline{24\text{ Nm}}$

Question 2

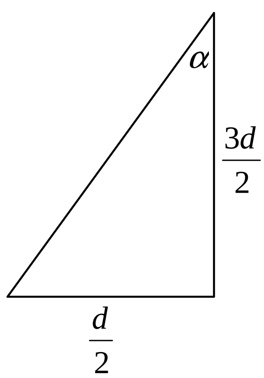
(i)

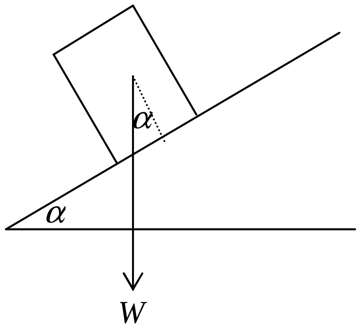


Since the lines of action of the weight and the friction are fixed, then using the 3 – force result, the line of action of the reaction passes through the intersection of the other two.

(ii) From symmetry, the centre of mass of a cylinder lies on the axis of the cylinder at a distance of half the vertical height i.e. a distance of $\frac{3d}{2}$ from its base.

For limiting equilibrium, the centre of mass must lie exactly vertically above the lowest point of contact.

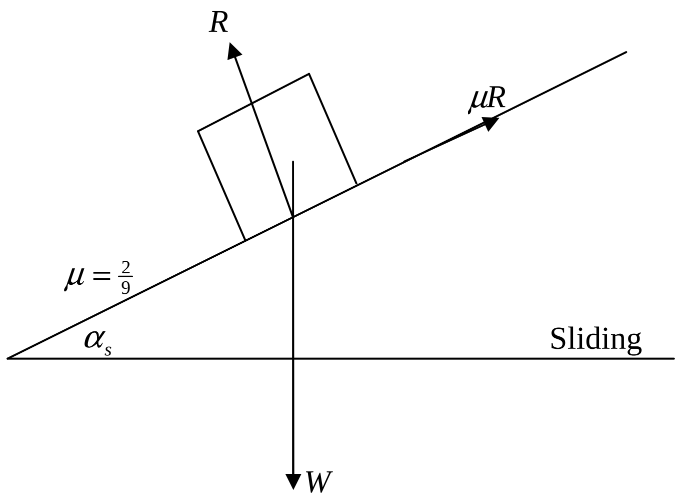




By referring to the diagram, the maximum angle occurs when

$$\tan\alpha = \frac{\frac{d}{2}}{\frac{3d}{2}} = \frac{1}{3} \qquad \Rightarrow \quad \alpha = \tan^{-1}\left(\frac{1}{3}\right) = \underline{18.4^0}$$

(iii)

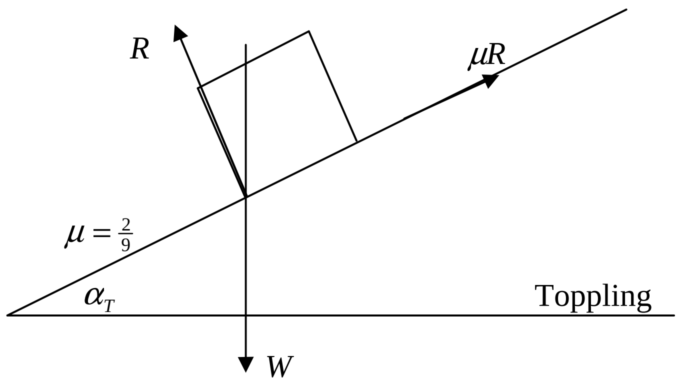


Resolving vertically: $R = W\cos\alpha_s \quad \ldots$(a)

Resolving horizontally: $\mu R = W\sin\alpha_s \quad \ldots$(b)

$$\frac{(b)}{(a)} \qquad \frac{\mu R}{R} = \frac{W\sin\alpha_s}{W\cos\alpha_s} \quad \Rightarrow \ \mu = \tan\alpha_s$$

$\therefore \ \tan\alpha_s = \dfrac{2}{9}$ for sliding

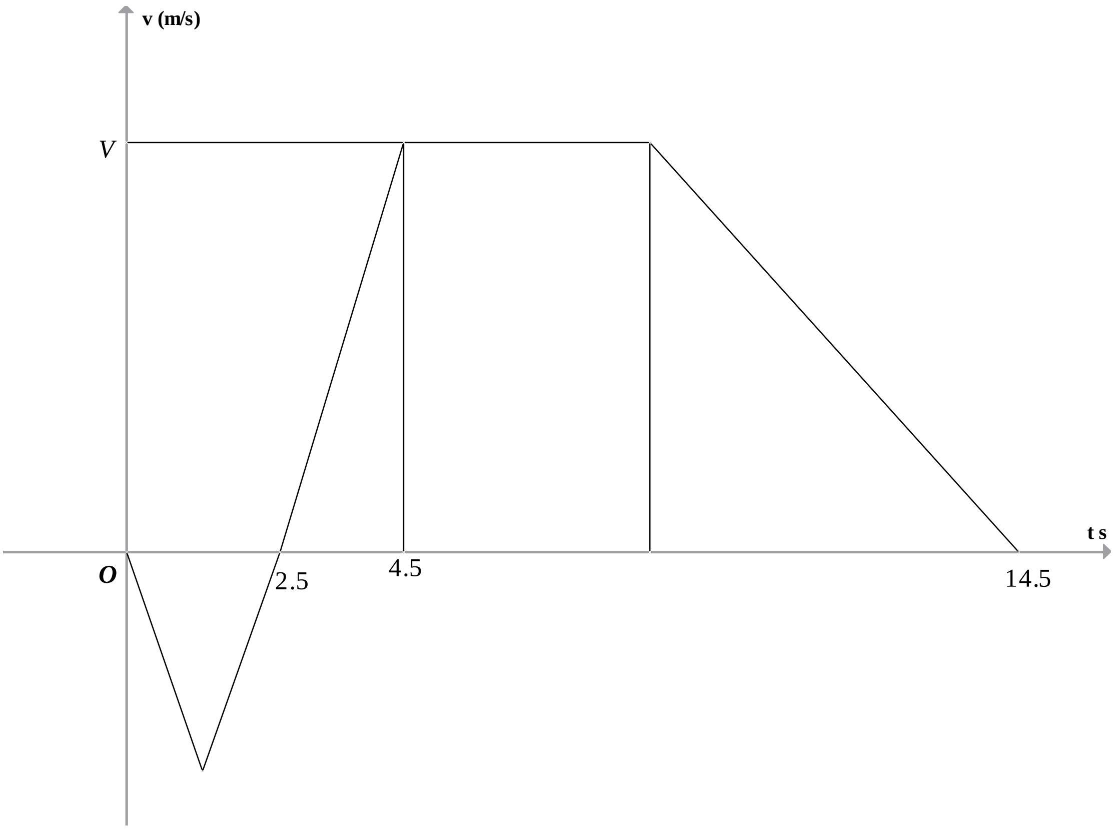


As in part (ii), $\tan\alpha_T = \dfrac{1}{3}$ for toppling

Since $\tan\alpha_s < \tan\alpha_T$ then $\alpha_s < \alpha_T$

$\Rightarrow$ sliding occurs first

Question 3



(i) From the graph *XA* = 4 m

$\Rightarrow$ area of triangle $= 4\text{m}$

Since the base of the triangle = 2.5 (time),

Then $4 = \frac{1}{2}(2.5)(V_{max})$, where V_{max} is the maximum speed for this section

$\Rightarrow \quad 8 = 2.5V_{max} \qquad \text{i.e.} V_{max} = \frac{8}{2.5} = \underline{3.2\text{m/s}}$

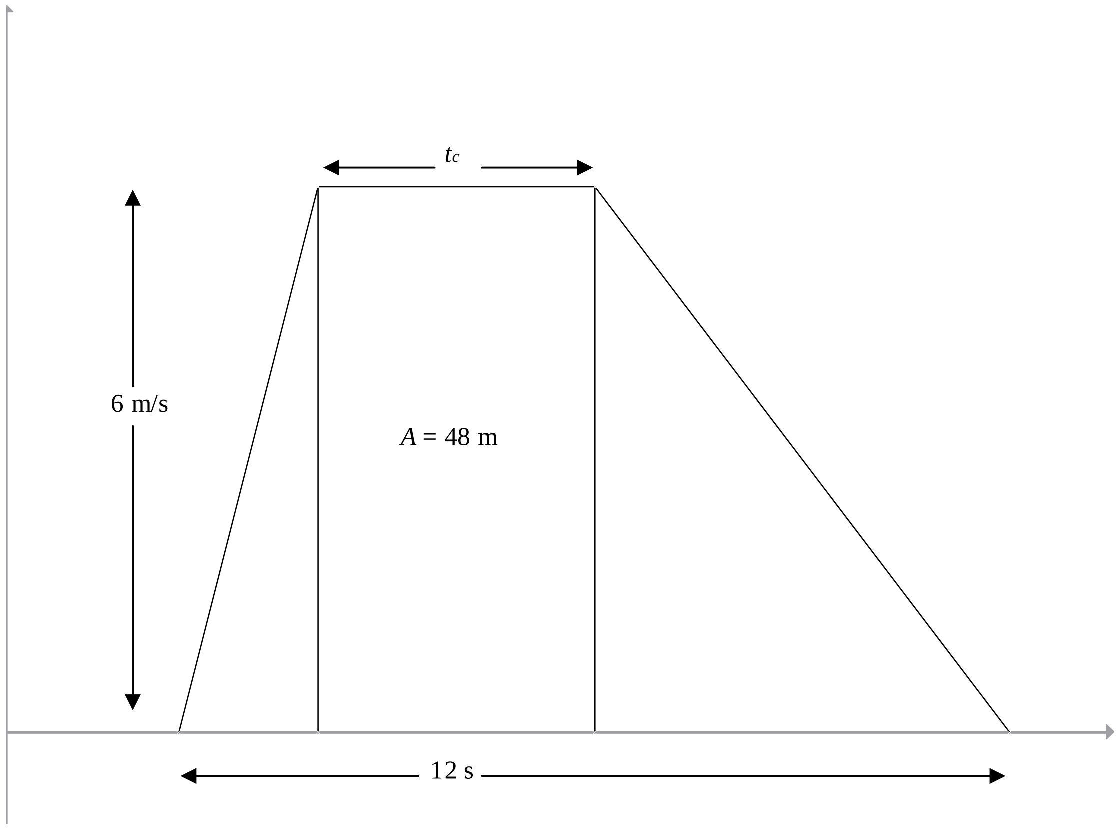
(ii) The gradient of a velocity – time graph is the acceleration

Thus Acceleration = $3\text{m/s}^2 = \frac{V}{\text{time}} = \frac{V}{2}$

$\Rightarrow \quad V = 3 * 2 = \underline{6\text{m/s}}$

(iii) Area of trapezium = $\frac{1}{2}(\text{sum of parallel sides})(\text{perpendicular height})$

$\therefore \quad 48 = \frac{1}{2}(t_c + 12) * 6$, by referring to the diagram below



$$\Rightarrow\ 48*2 = (t_c + 12)*6$$

$$\frac{48*2}{6} = t_c + 12 \quad \Rightarrow 16 = t_c + 12$$

i.e. t_c = 4 s

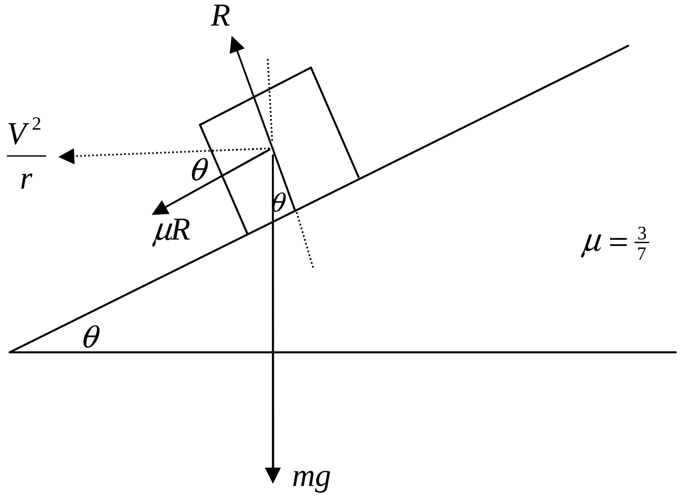
Thus P starts its deceleration at t = 4.5 + 4 = 8.5 s

(iv) The deceleration is found by finding the gradient of the last part

Deceleration = $\dfrac{0-6}{14.5-8.5} = \dfrac{-6}{6} = -1\,\text{m/s}^2$

$\therefore$ deceleration of 1 m/s^2

Question 4



Given $\tan\theta = \dfrac{7}{24}$

At equilibrium,

Resolving forces vertically: $R\cos\theta - \mu R\sin\theta = mg$

$$R(\cos\theta - \mu\sin\theta) = mg \quad \ldots\text{(a)}$$

Resolving forces horizontally: $F = ma = \dfrac{mv^2}{r}$

$$R\sin\theta + \mu R\cos\theta = \frac{mv^2}{r}$$

$$R(\sin\theta + \mu\cos\theta) = \frac{mv^2}{r} \quad \ldots\text{(b)}$$

$\dfrac{\text{(b)}}{\text{(a)}}$ $$\frac{R(\sin\theta + \mu\cos\theta)}{R(\cos\theta - \mu\sin\theta)} = \frac{\frac{mv^2}{r}}{mg}$$

$$\frac{(\sin\theta + \mu\cos\theta)}{(\cos\theta - \mu\sin\theta)} = \frac{v^2}{gr}$$

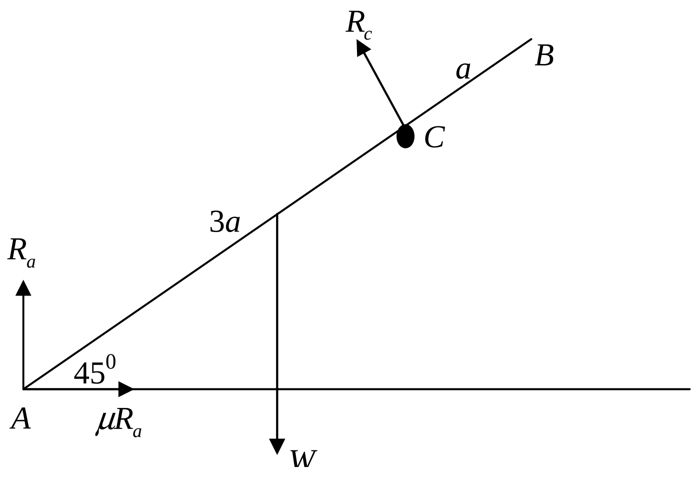
Dividing by $\cos\theta$: $$\frac{(\tan\theta + \mu)}{(1 - \mu\tan\theta)} = \frac{v^2}{gr}$$

$\Rightarrow$ $$v^2 = \frac{rg(\tan\theta + \mu)}{(1 - \mu\tan\theta)}$$

Substituting the given values $$v^2 = \frac{60(10)\left(\frac{7}{24} + \frac{3}{7}\right)}{\left(1 - \left(\frac{3}{7}\right)\left(\frac{7}{24}\right)\right)} = \frac{600\left(\frac{121}{168}\right)}{\frac{147}{168}} = 493.877$$

$\Rightarrow$ $$v = \sqrt{493.877} = \underline{22.22\text{m/s}}$$

Question 5



(i) Resolving horizontally: $\mu R_a = R_c \cos 45^0 = \dfrac{R_c}{\sqrt{2}}$ …(a)

Resolving vertically: $W = R_a + R_c \sin 45^0 = R_a + \dfrac{R_c}{\sqrt{2}}$ …(b)

Taking clockwise moments about A: $W \cos 45^0 (2a) = R_c (3a)$

$$\frac{W(2a)}{\sqrt{2}} = R_c (3a) \quad \Rightarrow \quad R_c = \frac{W(2a)}{3a\sqrt{2}} = \frac{W\sqrt{2}}{3} \text{ …(c)}$$

From (a) $R_a = \dfrac{R_c}{\mu\sqrt{2}} = \dfrac{1}{\mu\sqrt{2}}\dfrac{W\sqrt{2}}{3} = \dfrac{W}{3\mu}$, by using the result of (c)

Substituting this result and that of (c) in (b), we get:

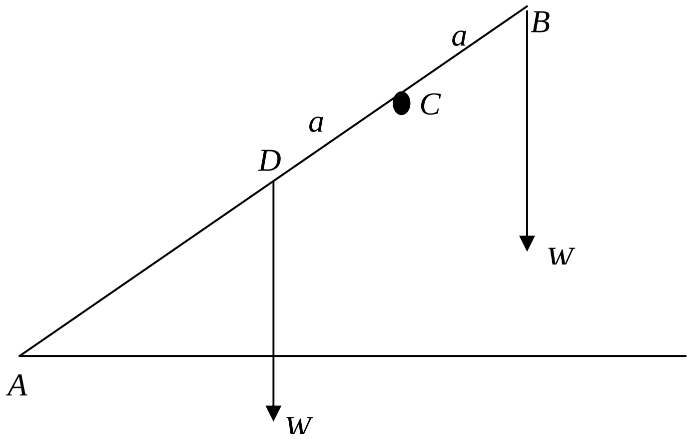
$$W = \frac{W}{3\mu} + \frac{1}{\sqrt{2}}\frac{W\sqrt{2}}{3} = \frac{W}{3\mu} + \frac{W}{3}$$

Dividing by W: $1 = \dfrac{1}{3\mu} + \dfrac{1}{3} \quad \Rightarrow \dfrac{2}{3} = \dfrac{1}{3\mu}$ i.e. $\mu = \underline{\dfrac{1}{2}}$

(ii) Referring to the diagram below:

The two weights at B and D are parallel and are equidistant from the peg C. Thus by symmetry their resultant is $2W$ (vertically downwards) and passes through C.

Thus if we consider their resultant instead of the 2 weights, we have 2 out of 3 forces in equilibrium whose line of action passes through C.



Using the 3 – force result, the total reaction S must pass through C as well.

$\therefore$ if λ is the angle of friction, then $\lambda = 45^0$

$\Rightarrow \quad \mu = \tan\lambda = \tan 45^0 = \underline{1}$

Question 6

(a) The pendulum beats every 2 s $\Rightarrow$ period T = $\underline{4 \text{ s}}$

The frequency $f = \dfrac{1}{T} = \dfrac{1}{4} = \underline{0.25\,\text{Hz}}$

Using $T = 2\pi\sqrt{\dfrac{l}{g}}$

Bringing l subject of the equation : $l = \dfrac{gT^2}{4\pi^2} = \dfrac{g(4)^2}{4\pi^2} = \underline{\dfrac{4g}{\pi^2}}$

(b) (i) When the particle is 0.06 m away from P, i.e. 0.3 – 0.06 = 0.24 m from the centre of motion, its speed is 0.9 m/s.

Using the formula: $v^2 = \omega^2(a^2 - x^2)$

Substituting $0.9^2 = \omega^2(0.3^2 - 0.24^2)$

$\Rightarrow \quad 0.9^2 = \omega^2(0.0324)$ i.e. $\omega^2 = \dfrac{0.9^2}{0.0324} = 25$

$\therefore \quad \omega = 5\text{ rad/s}$

The basic equation of S.H.M. is $\ddot{x} = -\omega^2 x$.

Substituting $\underline{\ddot{x} = -25x}$

(ii) the Period $T = \dfrac{2\pi}{\omega} = \underline{\dfrac{2\pi}{5}\text{ s}}$

(iii) Maximum velocity occurs when $x = 0$ (centre of motion)

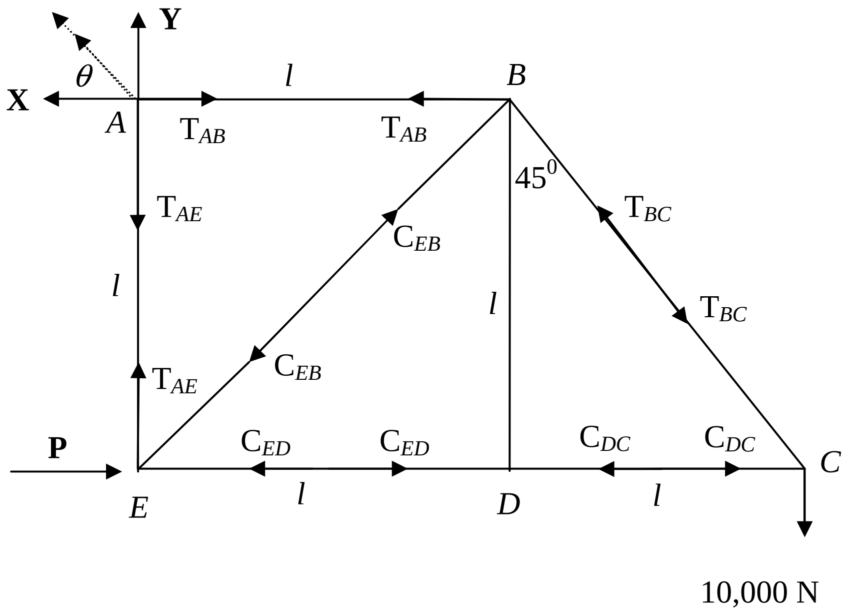
$\therefore\ v^2 = \omega^2 a^2 \ \Rightarrow\ v = \omega a = 5(0.3) = \underline{1.5\text{ m/s}}$

Maximum acceleration occurs when $x = a$ (extremes)

$\therefore\quad \ddot{x} = -\omega^2 x = -25(0.3) = -7.5\text{ m/s}^2$

Hence magnitude of acceleration = $\underline{7.5\text{ m/s}^2}$

Question 7



(i) Resolving vertically the whole system: **Y** = 10,000 N

Taking anticlockwise moments at A: **P**l = 10,000(2l).

$\Rightarrow$ **P** = $\underline{20{,}000\text{ N}}$

(ii) Resolving horizontally the whole system: **X** = **P** = 20,000 N

Let **R** be the reaction force on the framework at A.

$\mathbf{R} = \sqrt{\mathbf{X}^2 + \mathbf{Y}^2} = \sqrt{(20{,}000)^2 + (10{,}000)^2} = \underline{22{,}360.7\text{ N}}$

Let θ be the angle it makes with the horizontal,

Then $\tan\theta = \dfrac{10{,}000}{20{,}000} = \dfrac{1}{2} \quad \Rightarrow \quad \theta = \underline{26.57^0}$

(iii) From the diagram, it is clear that *BD* is neutral i.e. there is no force in *BD*

This can be easily verified because if we Resolve vertically at *D*: $T_{BD} = 0$

Resolving horizontally at *A*: $\mathbf{X} = T_{AB} \quad \Rightarrow \quad T_{AB} = \underline{20{,}000\text{ N}}$

Resolving vertically at *A*: $\mathbf{Y} = T_{AE} \quad \Rightarrow \quad T_{AE} = \underline{10{,}000\text{ N}}$

Resolving vertically at *C*: $10{,}000 = T_{BC}\cos 45^0 \quad \Rightarrow \quad T_{BC} = \underline{14142.1\text{ N}}$

Resolving horizontally at *C*: $T_{BC}\sin 45^0 = C_{DC}$

$$\Rightarrow \quad C_{DC} = 14142.1\sin 45^0 = \underline{10{,}000\text{ N}}$$

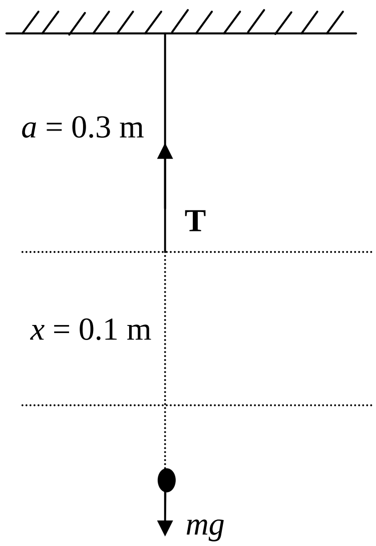
Resolving horizontally at *D*: $C_{ED} = C_{DC} \quad \Rightarrow \quad C_{ED} = \underline{10{,}000\text{ N}}$

Resolving vertically at *E*: $T_{AE} = C_{EB}\sin 45^0$

$$\Rightarrow \quad C_{EB} = \frac{T_{AE}}{\sin 45^0} = \frac{10{,}000}{\sin 45^0} = \underline{14{,}142.1\text{ N}}$$

In rods *AB*, *BC* and *EA*, there is tension, while in the rods *CD*, *DE* and *BE*, there is compression. There is no force in rod *BD*.

Question 8



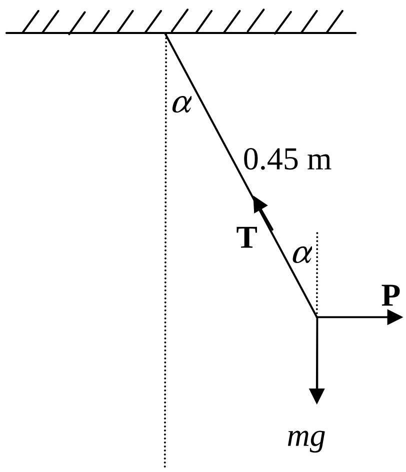
(i) $\mathbf{T} = \dfrac{\lambda x}{a}$

At equilibrium, resolving vertically: $\mathbf{T} = mg$

Given $a = 0.3$ m and $x = 0.1$ m

Substituting: $mg = \dfrac{\lambda(0.1)}{0.3} \quad \Rightarrow \lambda = \dfrac{0.3(mg)}{0.1} = \underline{3mg}$

(ii)



Using $\mathbf{T} = \dfrac{\lambda x}{a}$, we get $\mathbf{T} = \dfrac{3mg(0.45 - 0.3)}{0.3} = \dfrac{3mg}{2}$

At equilibrium, resolving vertically $\mathbf{T}\cos\alpha = mg$

Substituting $\dfrac{3\text{mg}}{2}\cos\alpha = mg \quad \Rightarrow \cos\alpha = \dfrac{2}{3} \quad \Rightarrow \alpha = \underline{48.2^0}$

(iii) At equilibrium, resolving horizontally $\mathbf{P} = \mathbf{T}\sin\alpha$

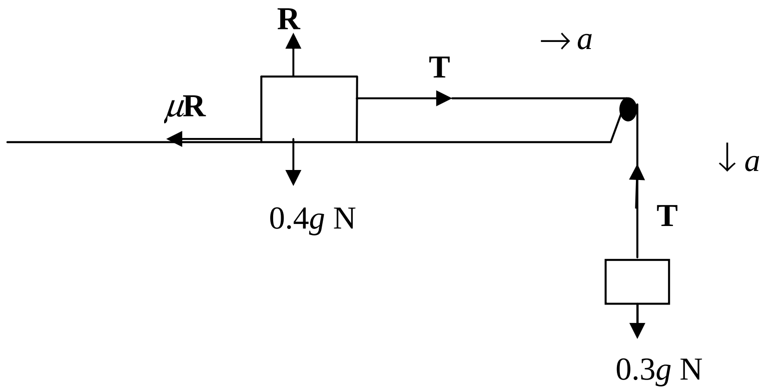
$$\Rightarrow \mathbf{P} = \frac{3mg}{2}\sin 48.2^0 = \underline{1.118mg}$$

Elastic Potential Energy = E.P.E. = $\dfrac{\lambda x^2}{2a} = \dfrac{3mg(0.45 - 0.3)^2}{2(0.3)}$

$$= \underline{0.1125mg}$$

Question 9

(i)



(ii) Resolving vertically **R** = 0.4g = 4 N

But the frictional force **F** = μ**R** = (0.5)(4) = <u>2 N</u>

(iii) Consider the 0.3 kg block

Resolving vertically and applying Newton's second law **F** = ma

$$0.3g - \mathbf{T} = 0.3a \quad \ldots(a)$$

Consider the 0.4 kg block

Resolving horizontally and applying Newton's second law **F** = ma

$$\mathbf{T} - \mu\mathbf{R} = 0.4a$$

$$\mathbf{T} - 2 = 0.4a \text{ by using part (ii)} \quad \ldots(b)$$

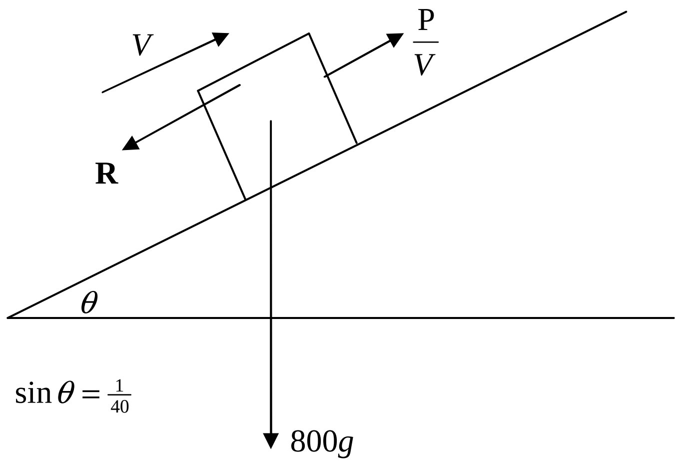
(a) + (b): $0.3g - 2 = 0.7a \Rightarrow a = \dfrac{0.3g - 2}{0.7} = \dfrac{10}{7}$ m/s^2

(iv) Applying $v = u + at$, we get $v = 0 + \left(\dfrac{10}{7}\right)7 = \underline{10 \text{ m/s}}$

(v) The 0.3 kg block would accelerate further. In fact its speed would increase at a faster rate since **T** = 0

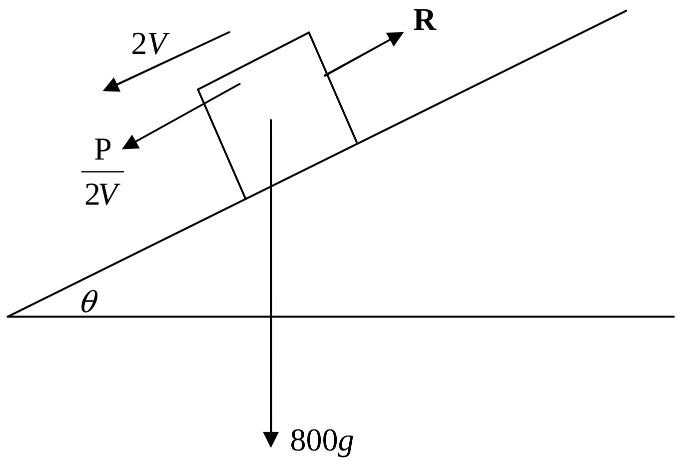
The 0.4 kg block would decelerate. In fact its speed would decrease since **T** = 0 and the only horizontal force acting on the particle would be the frictional force.

Question 10



(i) At equilibrium, resolving along the line of slope

$$\frac{\text{P}}{V} - \mathbf{R} = (800g\sin\theta) \quad \Rightarrow \quad \frac{10000}{V} - \mathbf{R} = 8000\left(\frac{1}{40}\right) = 200 \quad \ldots\text{(a)}$$



At equilibrium, resolving along the line of slope

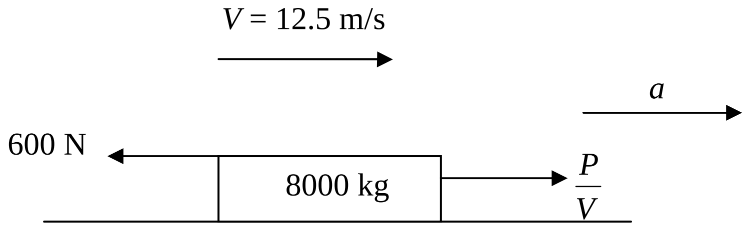
$$\mathbf{R} - \frac{\text{P}}{2V} = (800g\sin\theta) \quad \Rightarrow \quad \mathbf{R} - \frac{5000}{V} = 8000\left(\frac{1}{40}\right) = 200 \quad \ldots\text{(b)}$$

(a) + (b) $\quad \frac{10000}{V} - \frac{5000}{V} = 400 \quad \Rightarrow \quad \frac{5000}{V} = 400$

$$\text{i.e. } V = \frac{5000}{400} = \frac{25}{2} = \underline{12.5\text{ m/s}}$$

Substituting in (a): $\frac{10000}{12.5} - \mathbf{R} = 200 \quad \Rightarrow \quad \mathbf{R} = \underline{600\ \text{N}}$

(ii)



Applying Newton's 2nd law **F** = *ma*

$$\frac{P}{V} - 600 = 8000a$$

$$\frac{10000}{12.5} - 600 = 8000a \quad \Rightarrow 800 - 600 = 800a \quad \text{i.e. } a = \underline{0.25\ \text{m/s}^2}$$

Question 11

(i) Consider the horizontal axes:

$x = 36$ m (Given)

$\dot{x} = u\cos\alpha$, where u is the initial velocity and α is the angle of projection.

$$\dot{x} = 24\cos 30^0 = 24\left(\frac{\sqrt{3}}{2}\right) = 12\sqrt{3}$$

$$x = (u\cos\alpha)t = \left(12\sqrt{3}\right)t$$

$$\therefore\ 36 = \left(12\sqrt{3}\right)t \quad \Rightarrow \quad t = \frac{36}{12\sqrt{3}} = \frac{3}{\sqrt{3}} = \frac{3}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \underline{\sqrt{3}\ \text{s}}$$

(ii) Consider the vertical axes:

$s = h, \quad u = 24\sin 30^0, \quad a = -10 \text{ m/s}^2, \quad t = \sqrt{3}$

Using $s = ut + \frac{1}{2}at^2$

we get $h = (24\sin 30^0)(\sqrt{3}) + \frac{1}{2}(-10)(\sqrt{3})^2$

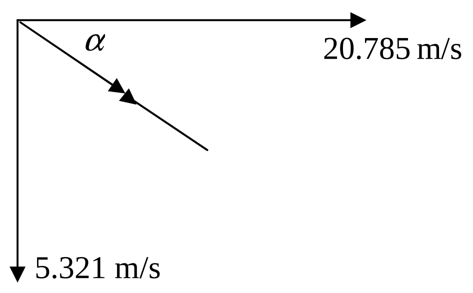
$h = 12(1.732) - 5(3) = \underline{5.785 \text{ m}}$

(iii) $\dot{x} = u\cos\alpha = 24\cos 30^0 = 20.785$

$\dot{y} = u\sin\alpha - gt = 24\sin 30^0 - 10(\sqrt{3}) = -5.321$

Speed = $\sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{(20.785)^2 + (-5.321)^2} = \underline{21.455 \text{ m/s}}$

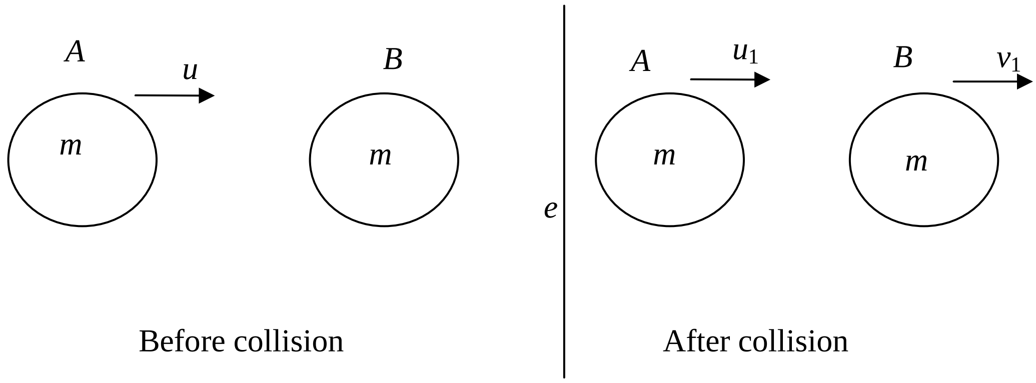
(iv)



$\tan\alpha = \frac{5.321}{20.785} = 0.256 \quad \Rightarrow \alpha = \underline{14.36^0}$ below the horizontal.

Question 12

(i) 1^{st} collision



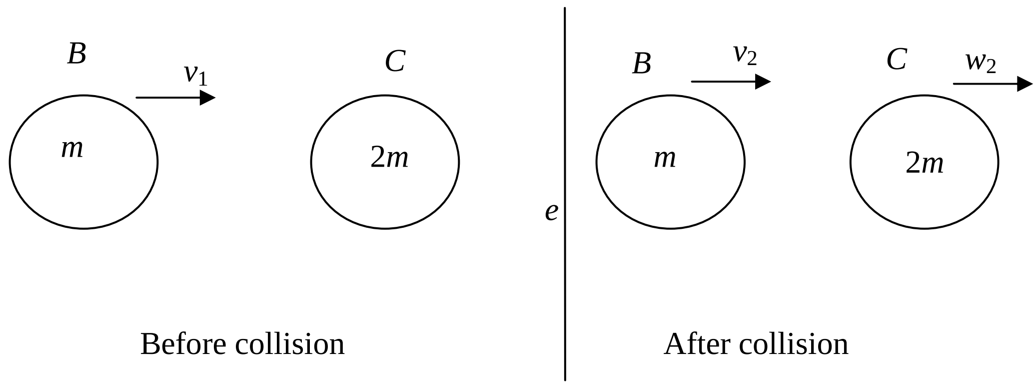
$$e = \frac{\text{Separation Speed}}{\text{Approach Speed}} = \frac{v_1 - u_1}{u - 0} \quad \Rightarrow \quad v_1 - u_1 = eu \quad \ldots(a)$$

Applying conservation of Momentum, we get

$$mu + m(0) = mu_1 + mv_1 \quad \Rightarrow \quad u_1 + v_1 = u \quad \ldots(b)$$

(a) + (b) $\quad 2v_1 = eu + u = u(1+e) \quad \Rightarrow \underline{v_1 = \frac{u}{2}(1+e)}$

(ii) 2^{nd} collision



$$e = \frac{w_2 - v_2}{v_1 - 0} \quad \Rightarrow \quad w_2 - v_2 = ev_1 \quad \ldots(c)$$

Applying conservation of Momentum, we get

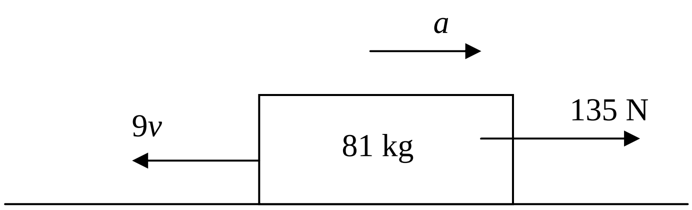
$$mv_1 + 2m(0) = mv_2 + 2mw_2 \quad \Rightarrow \quad v_2 + 2w_2 = v_1 \quad \ldots(d)$$

(c) + (d) $w_2 + 2w_2 = v_1(1+e) = \frac{u}{2}(e+1)(e+1)$

$\Rightarrow \quad 3\left(\frac{3}{8}u\right) = \frac{u}{2}(1+e)^2 \quad$ i.e. $\frac{9}{8} = \frac{(1+e)^2}{2}$

$(1+e)^2 = \frac{9}{4} \quad \Rightarrow (1+e) = \frac{3}{2} \quad$ i.e. $e = \underline{\frac{1}{2}}$

Question 13



(i) Applying Newton's second law **F** = *ma*, we get

$$135 - 9v = 81\frac{\mathrm{d}v}{\mathrm{d}t}$$

Divide by $135 - 9v$: $\quad 1 = \frac{81}{135-9v}\frac{\mathrm{d}v}{\mathrm{d}t}$

It simplifies to $\quad 1 = \frac{9}{15-v}\frac{\mathrm{d}v}{\mathrm{d}t}$

(ii) Integrating both sides of the differential equation

$$\int_0^t \mathrm{d}t = \int_0^v \frac{9}{15-v}\,\mathrm{d}v$$

$$t = -9\ln|15-v| \;\Big|_0^v = -9\ln|15-v| + 9\ln 15 = 9\ln\left|\frac{15}{15-v}\right|$$

$$\Rightarrow \quad \ln\left|\frac{15}{15-v}\right| = \frac{t}{9} \quad \text{i.e.} \quad \frac{15}{15-v} = e^{\frac{t}{9}}$$

$$15 = (15-v)e^{\frac{t}{9}} \quad \text{or} \quad 15e^{-\frac{t}{9}} = 15 - v$$

$$v = 15 - 15e^{-\frac{t}{9}} = \underline{15\left(1 - e^{-\frac{t}{9}}\right)}$$

As $t \to \infty \quad e^{-\frac{t}{9}} \to 0 \quad \therefore \; v = \underline{15\text{ m/s}}$

(iii) $v = \dfrac{\mathrm{d}x}{\mathrm{d}t} = 15(1 - e^{-\frac{t}{9}})$ by using part (ii)

Integrating this differential equation

$$\int_0^x \mathrm{d}x = \int_0^9 15(1 - e^{-\frac{t}{9}})\mathrm{d}t$$

$$\Rightarrow \quad x = \int_0^9 15(1 - e^{-\frac{t}{9}})\mathrm{d}t = \int_0^9 (15 - 15e^{-\frac{t}{9}})\mathrm{d}$$

$$x = \left[15t - 15\frac{e^{-\frac{t}{9}}}{-\frac{1}{9}}\right]_0^9 = \left[15t + 135e^{-\frac{t}{9}}\right]_0^9$$

$$x = 15(9) + 135e^{-1} - 15(0) - 135e^{0} = 135 + 135e^{-1} - 135 = \underline{49.66\text{ m}}$$

Question 14

(i) Given $v_{\mathrm{M}} = 6\mathbf{i} + 12\mathbf{j}$ and $v_{\mathrm{H}} = 12\mathbf{i} - 8\mathbf{j}$

The velocity of Harvey relative to Mario = $v_{H-M} = v_H - v_M$

$$= (12-6)\mathbf{i} + (-8-12)\mathbf{j} = \underline{(6\mathbf{i} - 20\mathbf{j})\text{ km/h}}$$

Given $r_{\mathrm{M}}(t=0) = 5\mathbf{i} - \mathbf{j}$ and $r_{\mathrm{H}}(t=0) = 18\mathbf{i} + 5\mathbf{j}$

The initial position of Harvey relative to Mario = $r_{H-M} = r_H - r_M$

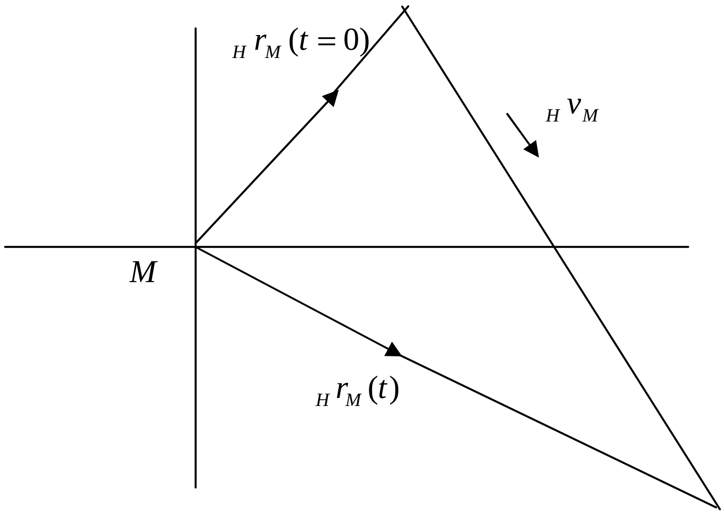
$$= (18-5)\mathbf{i} + (5+1)\mathbf{j} = \underline{(13\mathbf{i} + 6\mathbf{j})\text{ km}}$$

(ii) In general $r(t) = r(t=0) + tv$

$\therefore \; {}_H r_M(t) = {}_H r_M(t=0) + t\, {}_H v_M$

$\Rightarrow \quad {}_{\mathrm{H}} r_M(t) = = 13\mathbf{i} + 6\mathbf{j} + t(6\mathbf{i} - 20\mathbf{j}) = \underline{(13 + 6t)\mathbf{i} + (6 - 20t)\mathbf{j}\text{ km}}$

(iii) Diagram relative to M



Closest approach occurs when $_H r_M (t)$ is perpendicular to $_\mathrm{H} v_M (t)$

i.e. when $_H r_M (t) \cdot {}_\mathrm{H} v_M (t) = 0$

i.e. $\{(13 + 6t)\mathbf{i} + (6 - 20t)\mathbf{j}\} \cdot \{6\mathbf{i} - 20\mathbf{j}\} = 0$

$(13 + 6t)(6) + (6 - 20t)(-20) = 0$

$78 + 36t - 120 + 400t = 0$

$436t = 42 \quad \Rightarrow \quad t = \dfrac{42}{436} = 0.0963 \text{ hr} = \underline{5.78 \text{ mins}}$