University of Malta

Junior College

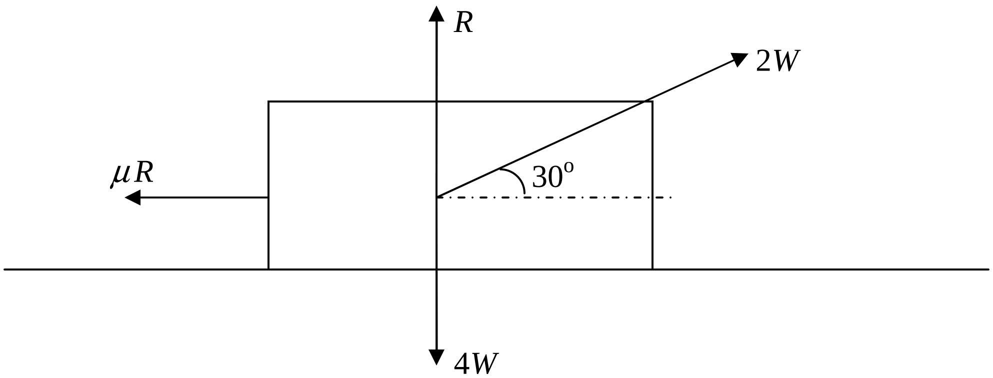
Subject: **Advanced Applied Mathematics**
Date: **June 2013**
Time: **9.00 - 12.00**

End of Year Test

Worked Solutions

Question 1

(a)



At Equilibrium:

Resolving vertically $R + 2W\sin 30^{\circ} = 4W$

$$R + 2W\left(\frac{1}{2}\right) = 4W$$

$$R + W = 4W \Rightarrow R = \underline{3W}$$

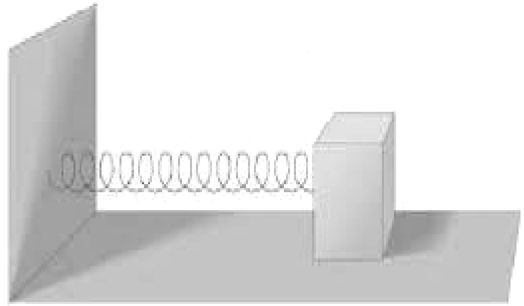
Resolving horizontally $\mu R = 2W\cos 30^{\circ}$

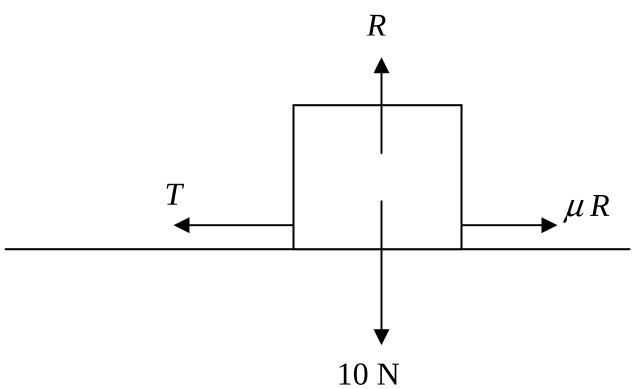
$$\mu(3W) = 2W\left(\frac{\sqrt{3}}{2}\right)$$

$$\Rightarrow \mu = \frac{\sqrt{3}}{3} = \underline{\frac{1}{\sqrt{3}}}$$

Angle of friction $= \lambda = \tan^{-1}(\mu) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \underline{30^{\circ}}$

(b)





With respect to the diagram above:

Resolving ⇳ $\quad R = 10\text{ N}$

$$\Leftrightarrow \quad T = \mu R = \left(\frac{1}{2}\right)(10) = 5\text{ N}$$

Hence $T = 5$ N

Applying Hooke's law i.e. $T = \dfrac{\lambda x}{a}$,

where a is the natural length = 20 cm = 0.2 m, λ is modulus of elasticity and x is the extension.

we get $$5 = \frac{50x}{0.2}$$

Hence $$x = \frac{5(0.2)}{50} = 0.02\text{ m}$$

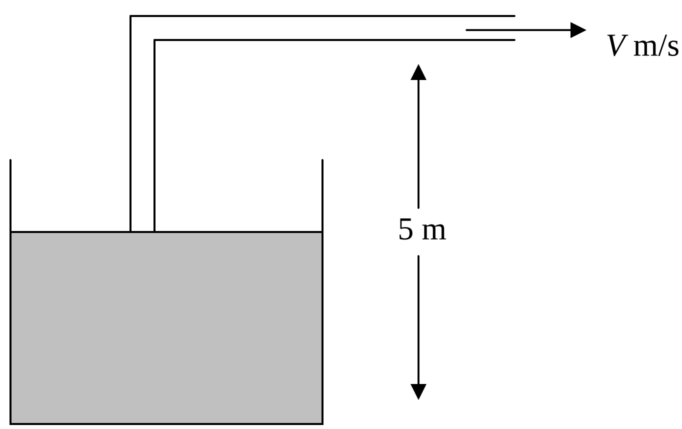
The above applies for the closest and furthest positions. In the former it is a compression, while in the latter it is an extension.

∴ Furthest point = 0.2 + 0.02 = 0.22 m from support.

Closest point = 0.2 – 0.02 = 0.18 m from support.

Question 2

(i)



The pump is 60% efficient. Hence the actual mechanical work is

60% of 0.825 kW = $\left(\frac{60}{100}\right) * 0.825 = 0.495\text{ kW} = 495\text{ W}$ …(a)

The volume of water per minute = 0.3 m^3.

Hence the volume per second $= \dfrac{0.3}{60} = \dfrac{1}{200}\ \text{m}^3 = 0.005\ \text{m}^3$

Since Density $= \dfrac{\text{Mass}}{\text{Volume}}$, then Mass = Density * Volume

$$= 1000\left(\frac{1}{200}\right) = 5\ \text{kg/s}$$

Power = Energy/s = K.E./s + P.E./s

$$\text{Energy/s} = \frac{1}{2}mv^2 + mgh$$

By (a) above, Energy/s = 495 J, m = 5 kg/s, h = 5 m

Substituting , we get $495 = \frac{1}{2}(5)v^2 + 5(10)5$

On simplifying $495 = 2.5v^2 + 250$

$\Rightarrow$ $495 - 250 = 2.5v^2$

Hence $v^2 = 98$ i.e. $v = \sqrt{98} = \underline{9.9\ \text{m/s}}$

(ii)

If V is the velocity of water through the nozzle and

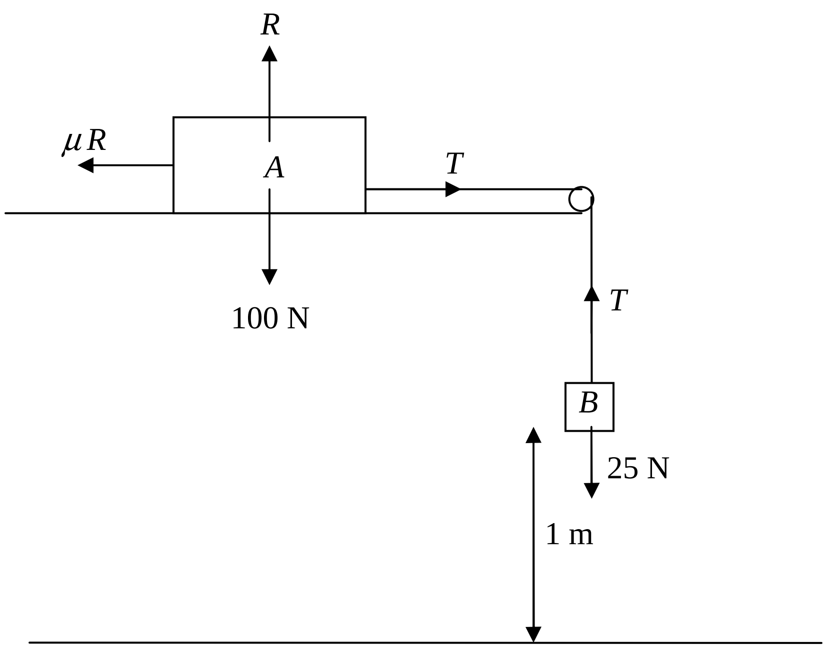
A is the cross sectional area of the nozzle,

Then volume per second = AV

$$0.005 = A(9.9)$$

Hence $A = \dfrac{0.005}{9.9} = 0.00051\,\text{m}^2 = \underline{5.1\,\text{cm}^2}$

Question 3



(i) Consider the block B.

Given $u = 0,\ t = 2,\ s = 1,\ a = ?$

Applying the equation of motion: $s = ut + \frac{1}{2}at^2$

$$1 = 0(2) + \frac{1}{2}a(2)^2$$

$$1 = \frac{1}{2}a(2)^2 \quad \Rightarrow \quad a = \underline{\frac{1}{2}\text{ m/s}^2}$$

Applying: $v = u + at$

$$v = 0 + \frac{1}{2}(2) \quad \Rightarrow \quad v = \underline{1\text{ m/s}}$$

(ii) Applying Newton's second law $F = ma$ on block B,

$$\downarrow \qquad 25 - T = 2.5\left(\frac{1}{2}\right)$$

$$25 - T = 1.25 \quad \Rightarrow \quad T = 25 - 1.25 = \underline{23.75\text{ N}}$$

At equilibrium,

Resolving $\Updownarrow$ on block A $\qquad R = 100\text{ N}$

Applying Newton's second law $F = ma$ on block A,

$$\rightarrow \qquad T - \mu R = 10a$$

Substituting $\qquad 23.75 - \mu(100) = 10\left(\frac{1}{2}\right)$

$$23.75 - \mu(100) = 5 \quad \Rightarrow \quad \mu = \underline{0.1875}$$

(iii) After B hits the floor, there is no tension T in the string.

Applying Newton's second law $F = ma$ on block A,

$$\leftarrow \qquad \mu R = 10a$$

Substituting $\qquad 0.1875(100) = 10a$

$$\Rightarrow \quad a = 1.875\text{ m/s}^2 \text{ decelerating}$$

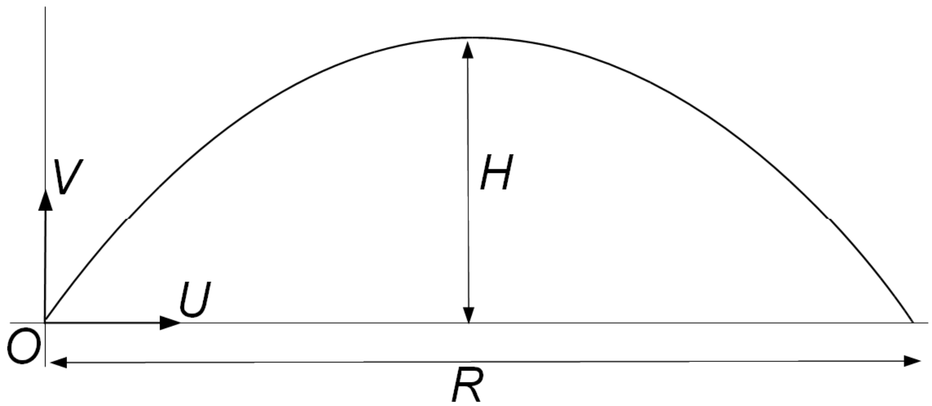
Applying the equation of motion: $v^2 = u^2 + 2as$

$$0 = (1)^2 + 2(-1.875)s$$

$$0 = 1 - 3.75s$$

Hence $\qquad s = \dfrac{1}{3.75} = \underline{0.267\text{ m}}$

Question 4



(a) Applying the equations of motion

$\uparrow \; s = 0$

$u = V$

$a = -g$

$\therefore$ substituting in $s = ut + \frac{1}{2}at^2$

$$0 = Vt - \frac{1}{2}gt^2$$

$$0 = t\left(V - \frac{1}{2}gt\right)$$

$\Rightarrow \quad t = 0, \text{ or } t = \frac{2V}{g}$

Considering in horizontal direction:

$\rightarrow \quad s = R$

$u = U$

$g = 0$

On substituting in the same equation of motion, we get $R = Ut$

Hence $R = U\left(\frac{2V}{g}\right) = \underline{\frac{2UV}{g}}$

For maximum height, time = $\frac{1}{2}$ time of flight = $\frac{1}{2}\left(\frac{2V}{g}\right) = \frac{V}{g}$

Applying the equations of motion for maximum height

$\uparrow \quad s = H$

$u = V$

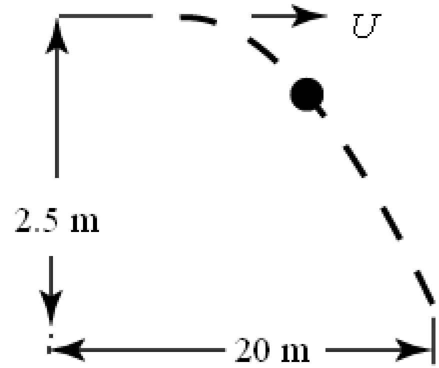
$a = -g$

$t = \frac{V}{g}$

$\therefore$ substituting in $\quad s = ut + \frac{1}{2}at^2$

$$H = V\left(\frac{V}{g}\right) - \frac{1}{2}g\left(\frac{V}{g}\right)^2 = \underline{\frac{V^2}{2g}}$$

(b) (i)



Let U be the initial speed

In the horizontal direction $\rightarrow$ $s = 20$

$u = U$

$a = 0$

$\therefore$ using $s = ut + \frac{1}{2}at^2$, we get $20 = Ut$ ….(i)

In the vertical direction $\downarrow$ $s = 2.5$

$u = 0$

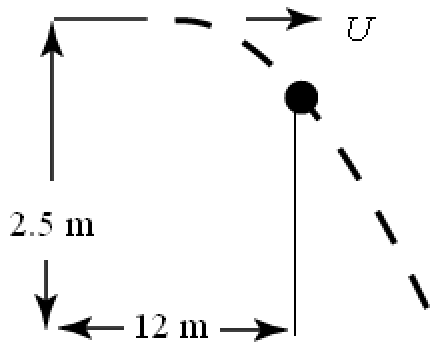
$a = 10$

$\therefore$ using $s = ut + \frac{1}{2}at^2$, we get $2.5 = 0 + \frac{1}{2}(10)t^2$

$2.5 = 5t^2$ i.e. $t^2 = \frac{1}{2}$ or $t = \frac{1}{\sqrt{2}}$

Substituting in (i), we get $20 = U\left(\frac{1}{\sqrt{2}}\right)$ $\Rightarrow$ $U = \underline{20\sqrt{2}\text{ m/s}}$

(ii)



In the horizontal direction $\rightarrow$ $s = 12$

$$u = U = 20\sqrt{2}$$

$$a = 0$$

$\therefore$ using $s = ut + \frac{1}{2}at^2$, we get $12 = 20\sqrt{2}\,t \Rightarrow t = \frac{12}{20\sqrt{2}} = \frac{3}{5\sqrt{2}}$

This is the time that the ball is over the net.

In the vertical direction $\downarrow$ $t = \frac{3}{5\sqrt{2}}$

$$u = 0$$

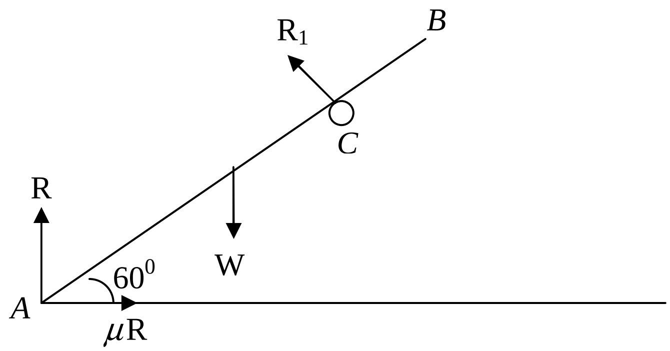
$$a = 10$$

$\therefore$ using $s = ut + \frac{1}{2}at^2$, we get $s = 0 + \frac{1}{2}(10)\left(\frac{3}{5\sqrt{2}}\right)^2 = 5\left(\frac{9}{50}\right) = \underline{0.9\text{ m}}$

This implies that the ball descended a distance of 0.9 m from the starting position, which is 25 m above ground.

$\Rightarrow$ the ball is 2.5 – 0.9 = <u>1.6 m</u> above ground.

Question 5



(i) Let R and R_1 be the reactions at points *A* and *C*, as shown.

Taking moments at *A* :

$$\frac{3l}{2}R_1 = lW\cos 60^0 \quad \Rightarrow \quad \frac{3l}{2}R_1 = \frac{lW}{2}$$

Thus $R_1 = \frac{W}{3}$

Resolving vertically, we get

$$\updownarrow \quad R + R_1\cos 60^0 = W \quad \Rightarrow \quad R + R_1\left(\frac{1}{2}\right) = W$$

Substituting the value of R_1, we get $R + \frac{W}{3}\left(\frac{1}{2}\right) = W$

Thus $R = W - \frac{W}{6} = \frac{5W}{6}$

(ii)

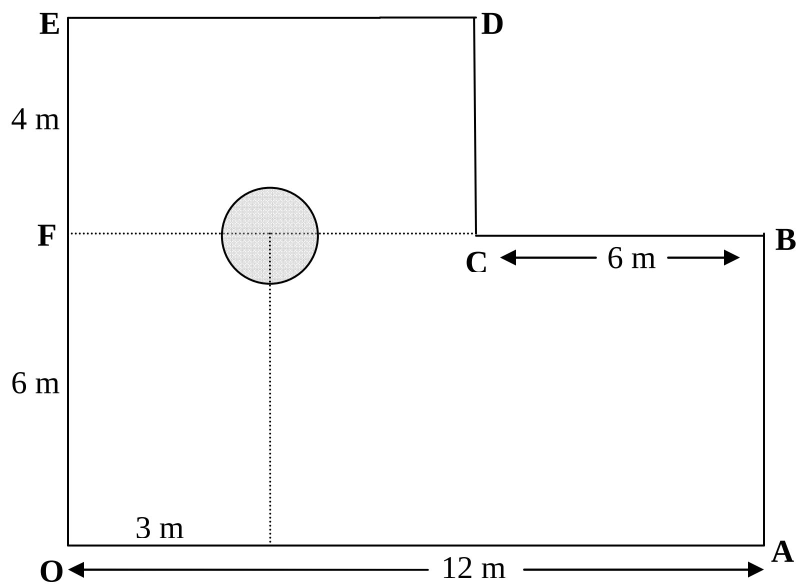
Resolving horizontally, we get

$$\leftrightarrow \quad \mu R = R_1\sin 60^0 \quad \Rightarrow \quad \mu R = R_1\left(\frac{\sqrt{3}}{2}\right)$$

Substituting the values of R and R_1, we get $\mu\left(\frac{5W}{6}\right) = \left(\frac{W}{3}\right)\left(\frac{\sqrt{3}}{2}\right)$

i.e. $\mu = \frac{\sqrt{3}}{5}$

Question 6



(i) The area of the shaded circle = $\pi\left(\sqrt{\frac{7}{\pi}}\right)^2 = 7$ sq.units

The area of the rectangle OABF = 12*6 = 72 sq.units

The area of the rectangle CDEF = 4*6 = 24 sq.units

Hence the area of cross section of the concrete block = 72 + 24 – 7

= 89 sq. units

Let M be the mass per unit area and

let $(\bar{x}, \bar{y})$ be the centre of mass of the block with respect to the origin O.

The centre of mass of the circle and rectangles OABF and CDEF are at (3, 6); (6, 3) and (3, 8) respectively.

Applying the principle of moments in the *x* direction, we get

$$89M\overline{gx} = 72Mg(6) + 24Mg(3) - 7Mg(3) = 483Mg$$

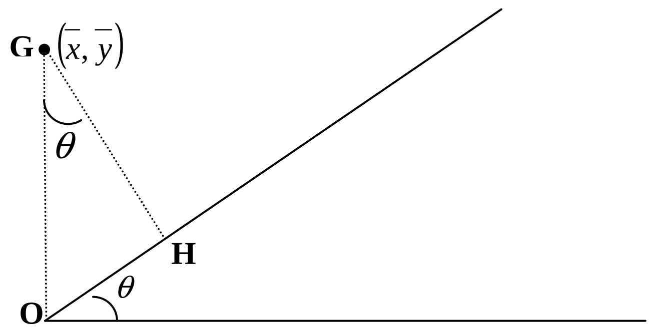
$$\Rightarrow \quad \bar{x} = \frac{483}{89} = \underline{5.43}$$

Applying the principle of moments in the y direction, we get

$$89\text{Mg}\bar{y} = 72\text{Mg}(3) + 24\text{Mg}(8) - 7\text{Mg}(6) = 366\text{Mg}$$

$$\Rightarrow \quad \bar{y} = \frac{366}{89} = \underline{4.11}$$

(ii)



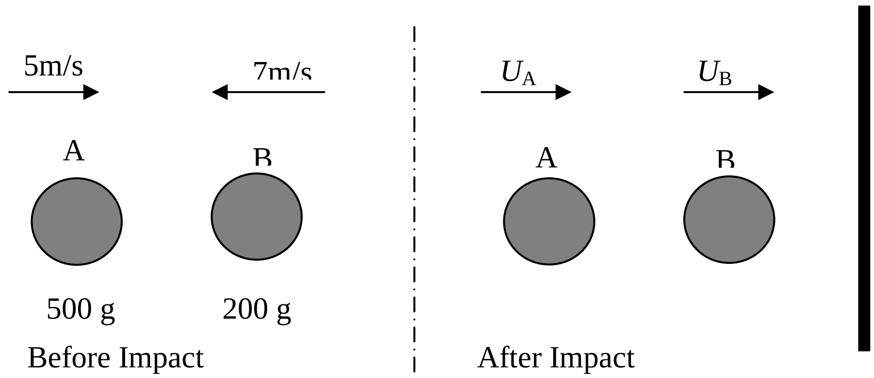
At equilibrium, the line through the centre of mass pass through O.

With respect to triangle OGH, where angle OHG is a right angle

$$\therefore \quad \tan\theta = \frac{\bar{x}}{\bar{y}} = \frac{5.43}{4.11} \quad \Rightarrow \quad \theta = \tan^{-1}\left(\frac{5.43}{4.11}\right) = \underline{52.9^0}$$

Question 7

(i)



Using the principle of conservation of momentum

$500(5) - 200(7) = 500U_A + 200\,U_B$

i.e. $1100 = 500U_A + 200\,U_B$

$11 = 5U_A + 2U_B \ldots$ (i)

Using the law of Restitution,

$\frac{3}{4} = \frac{U_B - U_A}{5-(-7)} = \frac{U_B - U_A}{12}$ i.e. $U_B - U_A = 9 \ldots$ (ii)

(i) $2U_B + 5U_A = 11$ –

(ii) ×2 $\underline{2U_B - 2U_A = 18}$

$7U_A = -7 \Rightarrow U_A = -1$ m/s

Thus sphere A changes direction and travels to the left.

Substituting this value in (ii), we get $U_B + 1 = 9$ i.e. $U_B = 8$m/s

Consider the impact of sphere B with the wall.

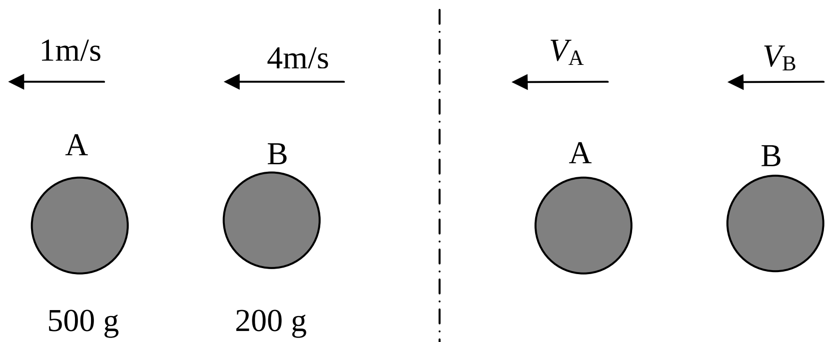
Using the law of Restitution, $\frac{1}{2} = \frac{U_B'}{U_B} \Rightarrow U_B' = \frac{U_B}{2} = \frac{8}{2} = 4$m/s

Sphere B travels in the direction of A with speed of 4m/s,

while sphere A travels with speed of 1m/s.

Thus there is surely going to be a second collision between the 2 spheres.

(ii)



Before 2nd Impact After 2nd Impact

$$500(1) + 200(4) = 500V_A + 200V_B$$

$$1300 = 500V_A + 200V_B \quad \Rightarrow \quad 13 = 5V_A + 2V_B \ \ldots \text{(i)}$$

Using the law of Restitution,

$$\frac{3}{4} = \frac{V_A - V_B}{4-(1)} = \frac{V_A - V_B}{3} \qquad \Rightarrow \quad V_A - V_B = \frac{9}{4} \ \ldots\text{(ii)}$$

(i) $\qquad 5V_A + 2V_B = 13 \quad -$

(ii) $\times 5$ $\qquad 5V_A - 5V_B = \dfrac{45}{4}$

$$7V_B = \frac{7}{4} \quad \Rightarrow V_B = \underline{\frac{1}{4}\text{ m/s}}$$

Substituting in (i): $\qquad 5V_A + 2\left(\dfrac{1}{4}\right) = 13 \quad \Rightarrow V_A = \underline{\dfrac{5}{2}\text{ m/s}}$

(iii) Initial K.E. $= \dfrac{1}{2}\left(\dfrac{500}{1000}\right)(5)^2 + \dfrac{1}{2}\left(\dfrac{200}{1000}\right)(7)^2 = 11.15\text{ J}$

Final K.E. $= \dfrac{1}{2}\left(\dfrac{500}{1000}\right)\left(\dfrac{5}{2}\right)^2 + \dfrac{1}{2}\left(\dfrac{200}{1000}\right)\left(\dfrac{1}{4}\right)^2 = 1.56875\text{ J}$

Loss in K.E. $= 11.15 - 1.56878 = \underline{9.58\text{ J}}$

Question 8

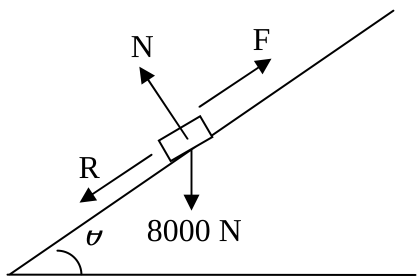
(i) Since Power = Force × Velocity

Then 10,000 = F × v, where v is the velocity of the car.

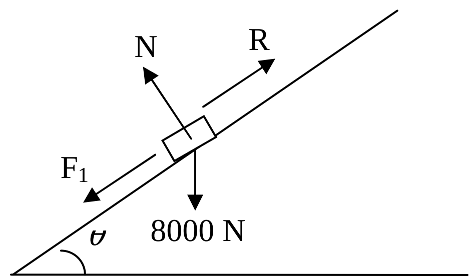
$$\Rightarrow \quad F = \frac{10000}{v}$$

When the car is going to descend, then 10,000 = F × 2v

$$\Rightarrow \quad F_1 = \frac{10000}{2v}$$



Ascent



Descent

Resolving forces along the line of greatest slope

↗ At equilibrium and car ascending $\frac{10000}{v} = R + 8000\sin\theta$

$$\frac{10000}{v} = R + 8000\left(\frac{1}{40}\right) = R + 200 \quad \ldots\text{(i)}$$

At equilibrium and car descending $\dfrac{10000}{2v} + 8000\sin\theta = R$

$$\frac{10000}{2v} + 8000\left(\frac{1}{40}\right) = R \quad \Rightarrow \quad R = \frac{10000}{2v} + 200 \quad \ldots\text{(ii)}$$

Substituting (ii) in (i), we get

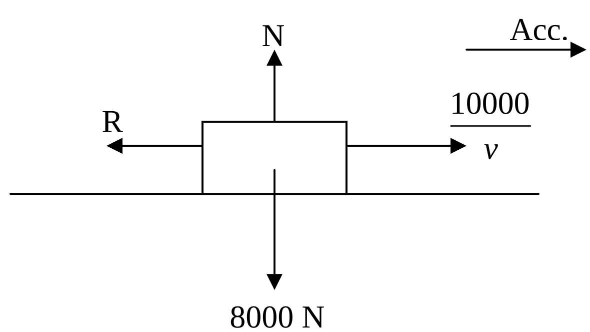
$$\frac{10000}{v} = \frac{10000}{2v} + 200 + 200$$

This reduces to $\dfrac{10000}{2v} = 400 \quad \Rightarrow \quad v = \dfrac{10000}{800} = \underline{12.5\text{m/s}}$

$\therefore$ speed of ascent = 12.5m/s

Substituting v in (i), we get $\dfrac{10000}{12.5} = R + 200 \quad \Rightarrow \quad R = \underline{600\text{ N}}$

(ii)



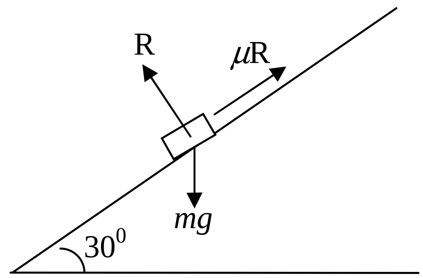
Applying Newton's second law i.e. $F = ma$, we get

$$\frac{10000}{v} - R = 8000a \quad \Rightarrow \quad \frac{10000}{12.5} - 600 = 800a$$

Hence $a = \underline{0.25\text{m/s}^2}$

Question 9

(a)



At equilibrium

Resolving perpendicular to the line of greatest slope

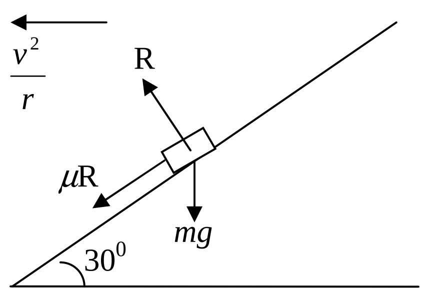
$$R = mg\cos 30^0 \quad \ldots(i)$$

Resolving along to the line of greatest slope

$$\mu R = mg\sin 30^0 \quad \ldots(ii)$$

(ii) ÷ (i) $\dfrac{\mu R}{R} = \dfrac{mg\sin 30^0}{mg\cos 30^0} \quad \Rightarrow \mu = \tan 30^0 = \underline{\dfrac{1}{\sqrt{3}}}$

(b)



At equilibrium

Resolving vertically: $R\cos 30^0 - \mu R\sin 30^0 = mg$

$\Rightarrow$ $R\left(\cos 30^0 - \mu\sin 30^0\right) = mg$ …(iii)

Resolving horizontally.

Using $F = ma = \dfrac{mv^2}{r}$ for motion in a circle

$\leftarrow$ $R\sin 30^0 + \mu R\cos 30^0 = \dfrac{mv^2}{r}$

$\Rightarrow$ $R\left(\sin 30^0 + \mu\cos 30^0\right) = \dfrac{mv^2}{r}$ …(iv)

(iv) $\div$ (iii) $\dfrac{R(\sin 30^0 + \mu\cos 30^0)}{R(\cos 30^0 - \mu\sin 30^0)} = \dfrac{\frac{mv^2}{r}}{mg} = \dfrac{v^2}{rg}$

It simplifies to $\dfrac{(\sin 30^0 + \mu\cos 30^0)}{(\cos 30^0 - \mu\sin 30^0)} = \dfrac{v^2}{60(10)}$

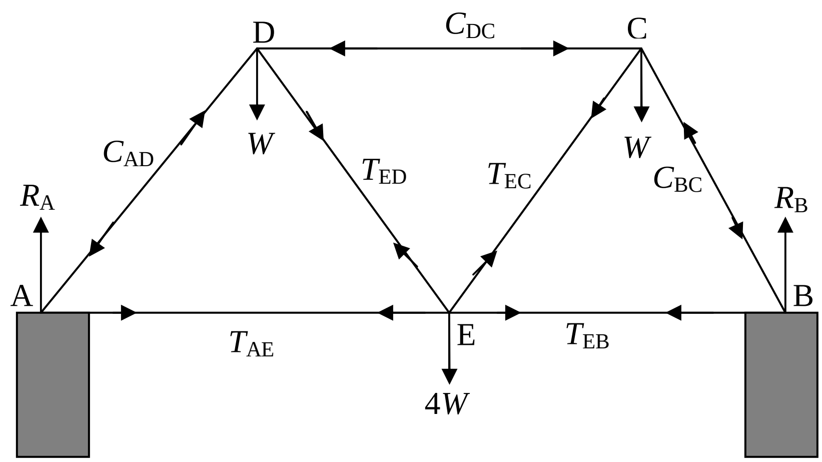
$$\frac{\left(0.5 + \frac{1}{\sqrt{3}}0.866\right)}{\left(0.866 - \frac{1}{\sqrt{3}}0.5\right)} = \frac{v^2}{600}$$

$\Rightarrow$ $v^2 = 1039.26$

i.e. $v =$ <u>32.24m/s</u>

Question 10

(i)



Since the 7 rods are identical, then all the triangles are identical

i.e. they are all equilateral triangles.

By symmetry $C_{AD} = C_{CB}$; $T_{DE} = T_{CE}$ and $T_{AE} = T_{EB}$

As the external forces are at equilibrium

By symmetry $R_A = R_B$

And resolving vertically, we have $R_A + R_B = W + W + 4W$

$\Rightarrow \quad 2R_B = 6W \quad$ or $\underline{R_B = R_A = 3W}$

(ii)

Consider the internal forces (Each joint is in equilibrium)

At A $\quad \updownarrow \quad R_A = C_{AD}\sin 60^0 \quad \Rightarrow \quad 3W = C_{AD}\left(\frac{\sqrt{3}}{2}\right)$ or $C_{AD} = \frac{6W}{\sqrt{3}}$

$\therefore C_{CB} = C_{AD} = \underline{\frac{6W}{\sqrt{3}}}$

At A $\quad \leftrightarrow \quad T_{AE} = C_{AD}\cos 60^0 \quad \Rightarrow \quad T_{AE} = \frac{6W}{\sqrt{3}}\left(\frac{1}{2}\right)$

$\therefore \; T_{EB} = T_{AE} = \underline{\frac{3W}{\sqrt{3}}}$

At E $\quad \updownarrow \quad 2T_{DE}\sin 60^0 = 4W \quad \Rightarrow 2T_{DE}\left(\frac{\sqrt{3}}{2}\right) = 4W$

$\therefore \; T_{EC} = T_{DE} = \underline{\frac{4W}{\sqrt{3}}}$

At D $\quad \leftrightarrow \quad C_{DC} = C_{AD}\cos 60^0 + T_{DE}\cos 60^0$

$$C_{DC} = \left(\frac{6W}{\sqrt{3}}\right)\left(\frac{1}{2}\right) + \left(\frac{4W}{\sqrt{3}}\right)\left(\frac{1}{2}\right) = \frac{5W}{\sqrt{3}}$$

Hence rods AE and EB – we have a tension of magnitude $\frac{3W}{\sqrt{3}}$

rods AD and BC – we have a compression of magnitude $\frac{6W}{\sqrt{3}}$

rods DE and EC – we have a tension of magnitude $\frac{4W}{\sqrt{3}}$

rod DC – we have a compression of magnitude $\frac{5W}{\sqrt{3}}$