

University of Malta

Junior College

Subject: **Advanced Applied Mathematics**
Date: **June 2014**
Time: **9.00 - 12.00**

End of Year Test

Worked Solutions

Question 1

(i) $3N$: $\underline{3\mathbf{i}}$ N

$$2N: 2\cos 60^\circ \mathbf{i} + 2\sin 60^\circ \mathbf{j}$$

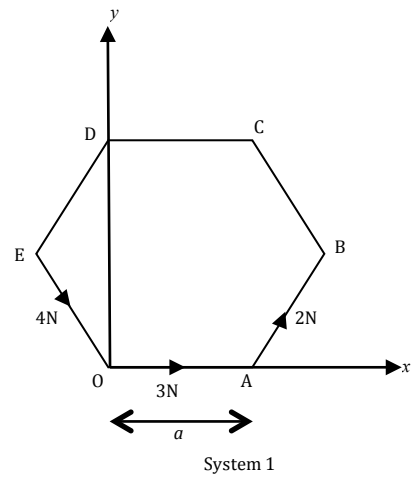
$$= \underline{(\mathbf{i} + \sqrt{3}\mathbf{j})} N$$

$$4N: 4\cos 60^\circ \mathbf{i} - 4\sin 60^\circ \mathbf{j}$$

$$= \underline{(2\mathbf{i} - 2\sqrt{3}\mathbf{j})} N$$

$$\text{Resultant } \sum \mathbf{F} = (3 + 1 + 2)\mathbf{i} + (\sqrt{3} - 2\sqrt{3})\mathbf{j}$$

$$= \underline{(6\mathbf{i} - \sqrt{3}\mathbf{j})} N$$



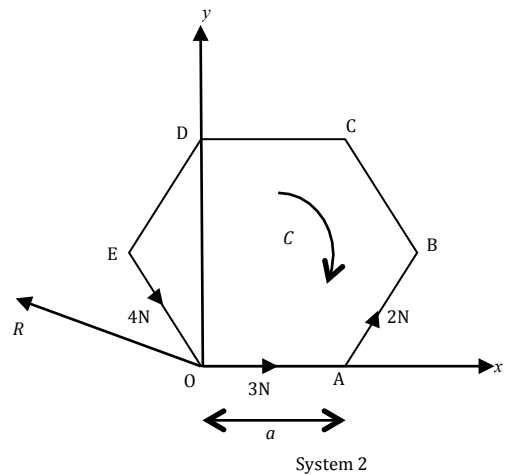
(ii) At equilibrium: $\sum (\mathbf{F} + \mathbf{R}) = 0$
 $\therefore \mathbf{R} = \underline{(-6\mathbf{i} + \sqrt{3}\mathbf{j})} N$

Also, taking moments about the origin for system:

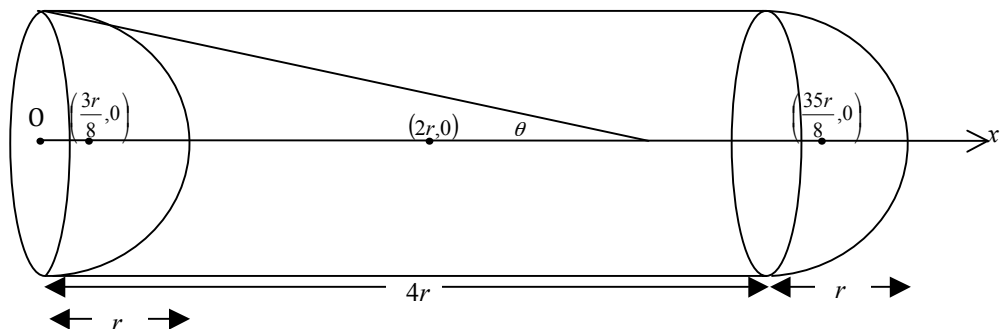
Moments about origin = 0

$$2\sin 60^\circ (a) = C$$

$$C = \underline{a\sqrt{3}} \text{ (Clockwise)}$$



Question 2



(i) The volume of the cylinder = $\pi r^2(4r) = 4\pi r^3$

The volume of the hemisphere = $\frac{2}{3}\pi r^3$

Let M be the density of the solid.

Mass of removed hemisphere: $\frac{2}{3}\pi r^3 M$

Mass of cylinder: $\pi r^2(4r)M = 4\pi r^3 M$

By symmetry, the centre of mass lies along the common axis of the 3 sections forming the solid.

The centre of mass of the cylinder = $2r$

The centre of mass of a hemisphere = $\frac{3r}{8}$ from the centre

By taking moments, we get:

$$4\pi r^3 M \bar{x} = 4\pi r^3 M(2r) + \frac{2}{3}\pi r^3 M\left(\frac{35r}{8}\right) - \frac{2}{3}\pi r^3 M\left(\frac{3r}{8}\right)$$

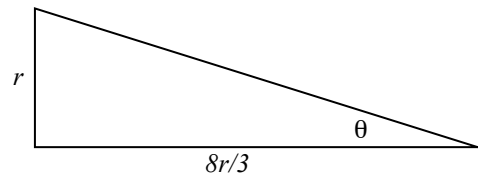
$$\bar{x} = \frac{4\pi r^3 M(2r) + \frac{2}{3}\pi r^3 M\left(\frac{35r}{8}\right) - \frac{2}{3}\pi r^3 M\left(\frac{3r}{8}\right)}{4\pi r^3 M}$$

Dividing top and bottom by $\pi r^3 M$:

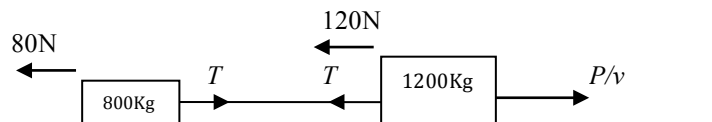
$$\bar{x} = \frac{8r + \frac{35r}{12} - \frac{r}{4}}{4} = \frac{8r}{3}$$

(ii) $\tan \theta = \frac{r}{\frac{8r}{3}} = \frac{3}{8}$

$\theta \approx \underline{20.6^\circ}$



Question 3

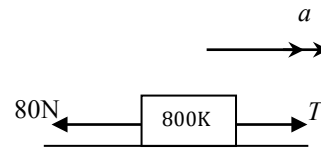


(i) For maximum acceleration $T = 2000\text{N}$.

$$F = ma$$

$$2000 - 80 = 800a$$

$$1920 = 800a \Rightarrow a = \underline{2.4\text{ms}^{-2}}$$



(ii) $v = 10\text{kmh}^{-1} = \frac{25}{9}\text{ms}^{-1}$

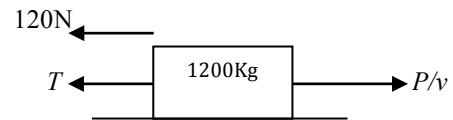
At maximum power, $T = 2000\text{N}$

$$F = ma, \text{ where } F \text{ (tractive force)} = \frac{P}{v}$$

$$\frac{P}{25/9} - 120 - 2000 = 1200(2.4)$$

$$\frac{9P}{25} = 2880 + 120 + 2000$$

$$\frac{9P}{25} = 5000 \Rightarrow P = \frac{25 \times 5000}{9} = 13888.89\text{W} \approx \underline{13.89\text{kW}}$$



Question 4

(i) Particle A:

Horizontally:

$$v = 14\text{ms}^{-1}, x = (14t)\text{m}$$

Vertically:

$$s = y$$

$$u = 0$$

$$v = ?$$

$$a = -10\text{ms}^{-2}$$

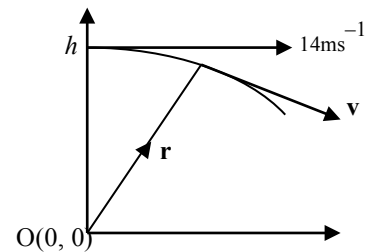
$$t = t$$

$$v = u + at$$

$$v = 0 - 10t$$

$$s = ut + \frac{1}{2}at^2$$

$$s = 0 - 5t^2$$



$$\therefore \mathbf{v} = (14\mathbf{i} - 10t\mathbf{j})\text{ms}^{-1}$$

$$\mathbf{r} = (14t\mathbf{i} + (h - 5t^2)\mathbf{j})\text{m}$$

Particle B:

Horizontally:

$$v = 17.5\text{ms}^{-1}, x = (17.5t)\text{m}$$

Vertically:

$$s = y$$

$$u = 0$$

$$v = ?$$

$$a = -10\text{ms}^{-2}$$

$$t = t$$

$$v = u + at$$

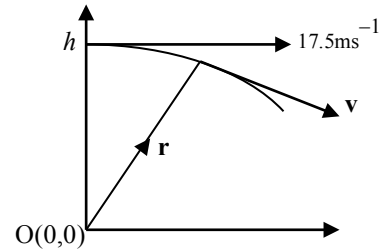
$$v = 0 - 10t$$

$$s = ut + \frac{1}{2}at^2$$

$$s = 0 - 5t^2$$

$$\therefore \mathbf{v} = (17.5\mathbf{i} - 10t\mathbf{j})\text{ms}^{-1}$$

$$\mathbf{r} = (17.5t\mathbf{i} + (h - 5t^2)\mathbf{j})\text{m}$$



(ii) Since the \mathbf{j} component of the position vector is the same for both particles, then they are at the same height throughout the flight, in particular when they hit the ground.

(iii) If they land 10m apart then the difference in the \mathbf{i} component of \mathbf{r} is 10m.

$$\text{i.e. } 17.5t - 14t = 10$$

$$t = \frac{10}{3.5} = 2.857\text{s}$$

So, time of flight = 2.857s.

Height of tower:

When particle hit the ground the \mathbf{j} component of $\mathbf{r} = 0$

$$\text{i.e. } h - 5t^2 = 0$$

$$h = 5t^2 \Rightarrow h = 5(2.857)^2 = \underline{\underline{40.82\text{ m}}}$$

Question 5

(i) Horizontally:

$$R_W = F_G$$

Vertically:

$$R_G = W$$

Taking moments about A:

$$W \cos \theta (a) = R_W \sin \theta (2a)$$

$$\frac{W}{\sqrt{5}} = R_W \cdot \frac{4}{\sqrt{5}} \Rightarrow R_W = \frac{W}{4}$$

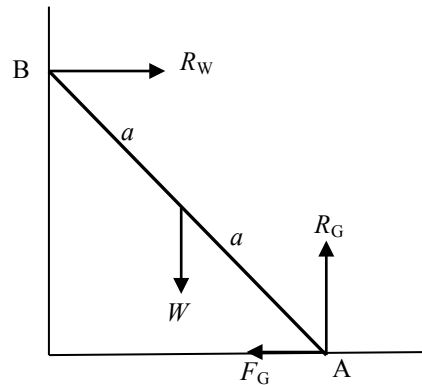
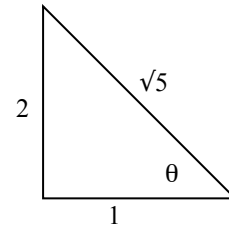
$$\therefore F_G = \frac{W}{4}$$

$$F_G \leq \mu R_G \Rightarrow \mu \geq \frac{F_G}{R_G} = \frac{W/4}{W} = \underline{\underline{\frac{1}{4}}}$$

$$\tan \theta = 2$$

$$\cos \theta = \frac{1}{\sqrt{5}}$$

$$\sin \theta = \frac{2}{\sqrt{5}}$$



(ii) Limiting equilibrium:

Vertically:

$$R_G = 2W$$

Horizontally:

$$R_W = F_G = \frac{5}{16}(2W) = \frac{5W}{8}$$

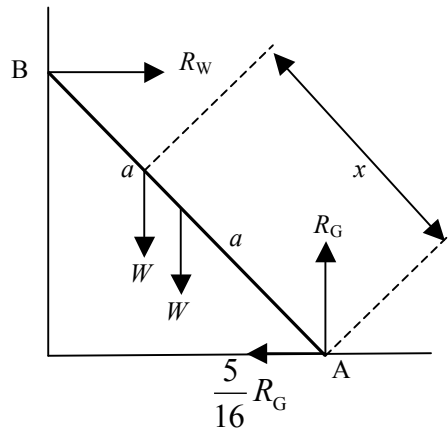
Taking moments about A:

$$W \cos \theta (a) + Wx \cos \theta = R_W \sin \theta (2a)$$

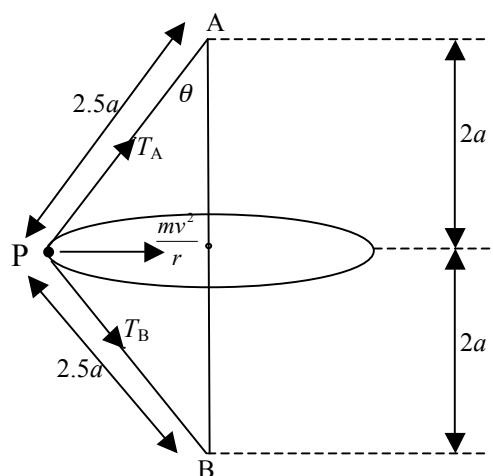
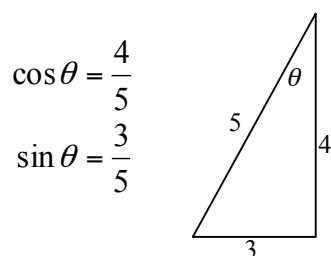
$$\frac{W}{\sqrt{5}}(a) + \frac{W}{\sqrt{5}}(x) = \frac{5W}{8} \left(\frac{2}{\sqrt{5}} \right) 2a$$

$$a + x = \frac{5}{2}a$$

$$x = \frac{5}{2}a - a = \underline{\underline{\frac{3a}{2}}}$$



Question 6



(i) Vertical equilibrium:

$$T_A \cos \theta = T_B \cos \theta + mg$$

$$(T_A - T_B) \frac{4}{5} = mg \Rightarrow T_A - T_B = \frac{5}{4} mg \dots (1)$$

Horizontally:

$$F = ma = mr\omega^2$$

$$T_A \sin \theta + T_B \sin \theta = mr\omega^2$$

$$(T_A + T_B) \left(\frac{3}{5} \right) = m(2.5a \sin \theta) \omega^2 = m \left(2.5 \times \frac{3}{5} \right) a \omega^2$$

since from diagram $r = 2.5a \sin \theta$

$$T_A + T_B = \frac{5}{2} ma \omega^2 \dots (2)$$

(1) + (2):

$$2T_A = \frac{5}{2} ma \omega^2 + \frac{5}{4} mg$$

$$T_A = \frac{1}{2} \left(\frac{5}{2} ma \omega^2 + \frac{5}{4} mg \right)$$

$$T_A = \frac{5}{4} ma \omega^2 + \frac{5}{8} mg = \underline{\underline{\frac{5m}{8} (2a\omega^2 + g)}}$$

(2) - (1):

$$2T_B = \frac{5}{2}ma\omega^2 - \frac{5}{4}mg$$

$$T_B = \frac{5}{4}ma\omega^2 - \frac{5}{8}mg = \underline{\underline{\frac{5m}{8}(2a\omega^2 - g)}}$$

(ii) For minimum ω , $T_B = 0$.

$$2a\omega_{\min}^2 = g \Rightarrow \omega_{\min}^2 = \frac{g}{2a}$$

$$\therefore \underline{\underline{\omega_{\min}^2 > \frac{g}{2a}}}$$

Question 7

Those forces marked with an asterisk are equal by symmetry.

(i) For the whole system, resolving horizontally:

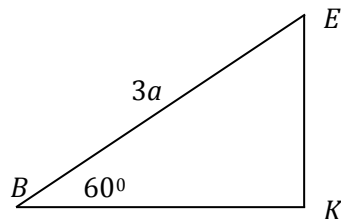
$$X_A = 0 \text{ (one horizontal force)}$$

Vertically:

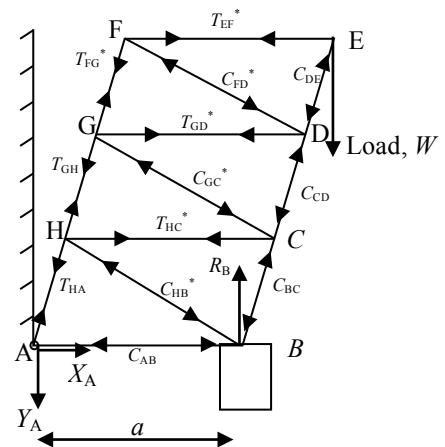
$$Y_A + W = R_B$$

Since the rods are identical, then they are all equal and each triangle is equilateral.

Hence the rod $BE = 3a$, and it makes an angle of 60° with the horizontal.



With respect to the above diagram $BK = 3a \cos 60^\circ = \frac{3a}{2}$



Taking moments about B:

$$Y_A(a) = W \frac{3a}{2} \Rightarrow Y_A = \frac{3W}{2}$$

The reaction at A is simply a vertical downward force of magnitude $\frac{3W}{2}$. Also, this means that actually there is no need for a hinge. A simple support or peg from above will be enough.

(ii) Resolving vertically at A:

$$T_{AH} \sin 60^\circ = Y_A = \frac{3W}{2} = \frac{3}{2}(2\sqrt{3}) = 3\sqrt{3}\text{MN} \Rightarrow T_{AH} = 6\text{MN}$$

Horizontally:

$$C_{AB} = T_{AH} \cos 60^\circ = T_{AH} \left(\frac{1}{2}\right) = 3\text{MN}$$

Resolving vertically at E:

$$C_{DE} \sin 60^\circ = 2\sqrt{3} \Rightarrow C_{DE} = 4\text{MN}$$

Horizontally:

$$T_{EF} = C_{DE} \cos 60^\circ = 2\text{MN}$$

$$\text{But, } T_{EF} = T_{FG} = C_{DF} = T_{GD} = C_{GC} = T_{HC} = C_{HB} = 2\text{MN}$$

Resolving vertically at D:

$$C_{CD} \sin 60^\circ = C_{DF} \sin 60^\circ + C_{DE} \sin 60^\circ \Rightarrow C_{CD} = 2 + 4 = 6\text{MN by using previous results}$$

Resolving vertically at G:

$$T_{GH} \sin 60^\circ = T_{FG} \sin 60^\circ + C_{CG} \sin 60^\circ \Rightarrow T_{GH} = 2 + 2 = 4\text{MN by above}$$

Resolving vertically at C:

$$C_{BC} \sin 60^\circ = C_{GC} \sin 60^\circ + C_{CD} \sin 60^\circ \Rightarrow C_{BC} = 2 + 6 = 8\text{MN by above}$$

Summary:

EF, FG, GD, HC: 2MN in tension.

FD, GC, HB: 2MN in compression.

AB: 3MN in compression.

DE: 4MN in compression.

CD: 6MN in compression.

GH: 4MN in tension.

AH: 6MN in tension.

BC: 8MN in compression.

Question 8

(i) At A:

$$PE = KE = EPE = 0 \rightarrow \text{Total Energy} = 0$$

At B:

$$PE = -mg(3a) = -3mga$$

$$KE = 0$$

$$EPE = \frac{\lambda x^2}{2a} = \frac{\lambda(2a)^2}{2a} = 2a\lambda$$

$$\text{Therefore, Total Energy} = 2a\lambda - 3mga$$

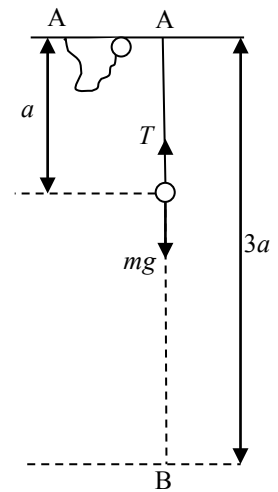
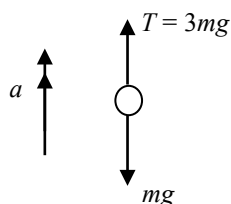
By the principle of conservation of energy:

$$2a\lambda - 3mga = 0$$

$$\Rightarrow \lambda = \frac{3mg}{2}$$

(ii) At B:

$$T = \frac{\lambda x}{a} = \frac{\frac{3mg}{2}(2a)}{a} = 3mg$$

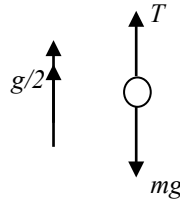


Applying $F = ma$ vertically up:

$$\begin{aligned}F &= ma \\3mg - mg &= ma \\a &= \underline{2g}\end{aligned}$$

(iii) Applying $F = ma$ vertically up:

$$\begin{aligned}T - mg &= \frac{mg}{2} \\T &= \frac{3mg}{2}\end{aligned}$$



Hooke's law:

$$\begin{aligned}T &= \frac{\lambda x}{a} \Rightarrow \frac{3mg}{2} = \frac{3mg}{2} \frac{x}{a} \\ \therefore x &= a\end{aligned}$$

Let this point be C. Then at C:

$$\text{KE} = \frac{1}{2}mv^2$$

$$\text{PE} = -mg(2a) = -2mga$$

$$\text{EPE} = \frac{\lambda x^2}{2a} = \frac{\left(\frac{3mg}{2}\right)a^2}{2a} = \frac{3mga}{4}$$

$$\text{Therefore, Total Energy} = \frac{1}{2}mv^2 + \frac{3mga}{4} - 2mga$$

Applying the principle of conservation of energy

$$\frac{1}{2}mv^2 + \frac{3mga}{4} - 2mga = 0$$

$$\frac{1}{2}mv^2 - \frac{5mga}{4} = 0$$

$$v^2 = \frac{5ga}{2}$$

$$v = \sqrt{\frac{5ga}{2}}$$

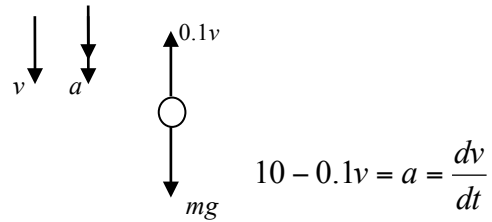
Question 9

(i) Applying $F = ma$ horizontally down:

$$F = ma$$
$$mg - 0.1v = ma$$

Substituting $m = 1, g = 10 \rightarrow$

$$\text{Hence, } \underline{\underline{\frac{dv}{dt} = 10 - 0.1v}}}$$



(ii) $\int_{v=0}^v \frac{dv}{10 - 0.1v} = \int_{t=0}^t dt$

$$-\frac{1}{0.1} \ln|10 - 0.1v|_0^v = t \Big|_0^t$$

$$-10(\ln|10 - 0.1v| - \ln 10) = t - 0$$

$$10 \ln \left| \frac{10}{10 - 0.1v} \right| = t$$

$$\ln \left| \frac{10}{10 - 0.1v} \right| = 0.1t$$

$$\frac{10}{10 - 0.1v} = e^{0.1t}$$

$$10 - 0.1v = 10e^{-0.1t}$$

$$0.1v = 10(1 - e^{-0.1t})$$

$$\underline{\underline{v = 100(1 - e^{-0.1t})}}$$

(iii) As $t \rightarrow \infty$, $e^{-0.1t} \rightarrow 0$

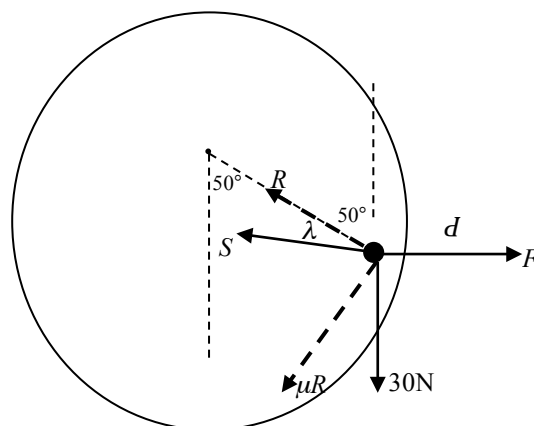
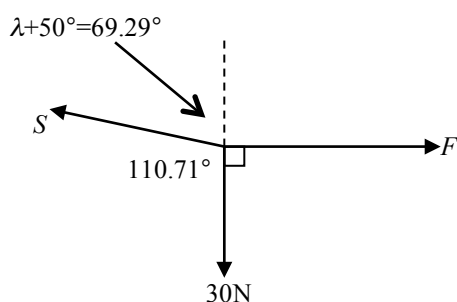
$$\therefore \lim_{t \rightarrow \infty} v = \underline{\underline{100 \text{ ms}^{-1}}}$$

Question 10

- (i) S is the resultant of the reaction R and the friction μR .
 λ is the angle of friction.

$$\mu = 0.35 \Rightarrow \lambda = \tan^{-1} 0.35 \approx 19.29^\circ$$

- (iii) The diagram of forces hence becomes...



Using Lami's Theorem:

$$\frac{30}{\sin 159.29^\circ} = \frac{F}{\sin 110.71^\circ}$$

$$F = \frac{30 \sin 110.71^\circ}{\sin 159.29^\circ} = \underline{79.35 \text{ N}}$$

Using once more Lami's Theorem:

$$\frac{S}{\sin 90^\circ} = \frac{30}{\sin 159.29^\circ}$$

$$\text{Hence } S = \frac{30}{\sin 159.29^\circ} = \underline{84.83 \text{ N}}$$

S makes an angle of $\underline{110.71^\circ}$ with the 30 N force.
